April 28, 2010

Five-dimensional AGT Relation and the Deformed β-ensemble

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Abstract

We discuss an analog of the AGT relation in five dimensions. We define a q-deformation of the β-ensemble which satisfies q-$\mathcal{W}_N$ constraint. We also show a relation with the Nekrasov partition function of 5D $SU(N)$ gauge theory with $N_f = 2N$.

1 Introduction

In [1], Alday Gaiotto and Tachikawa discovered remarkable relations between the 4D $\mathcal{N}=2$ super conformal gauge theories and the 2D Liouville CFT. Some explanations have been addressed from β-ensemble (generalized matrix model) [2][3] in [4]–[7].

In the pure $SU(2)$ case, the AGT relation [8] between the instanton part of the partition functions of the gauge theory and correlation functions of the Virasoro algebra is extended naturally to 5D in [9].[1] The instanton counting [10]–[13] of the 5D gauge theory [14] can be viewed as a q-analog of 4D cases [15]–[17], and there also exists a natural q-deformation of the Virasoro/$\mathcal{W}_N$ algebra [18]–[21].

In this article, we will study a 5D extension of the AGT relation with $N_f=2N$. The A_{N-1} type quiver matrix model (the ITEP model) [22] was generalized as a β-ensemble [2] satisfying the $\mathcal{W}_N$ constraint by [3]. Under the strategy of [3], we will introduce q-deformed β-ensemble which automatically satisfies q-$\mathcal{W}_N$ constraint and show a relation with the 5D Nekrasov partition function of $SU(N)$ gauge theory with $N_f=2N$.

This paper is organized as follows: In section 2, we start with recapitulating the result of the q-$\mathcal{W}_N$ algebra and also define primary fields. In section 3, we introduce q-deformed β-ensemble which automatically satisfies q-$\mathcal{W}_N$ constraint. Section 4 deals with the $N=2$ case. Finally in section 5, we explain a reduction of the 5D Nekrasov partition function to the q-hypergeometric function and show a coincidence with the partition function of our q-deformed β-ensemble. Appendix A contains a definition of the Macdonald polynomial and several useful formulas. Proof of the key equation is shown in appendix B. Relations with the Kaneko's integral formula and the Jackson integral formulas are given in appendices C and D, respectively. In appendix E, we review the 4D case. Appendix F is devoted to a list of notations for bosons.

Notation. Let $[\,n\,]_p := (p^{\frac{n}{2}}-p^{-\frac{n}{2}})/(p^{\frac{1}{2}}-p^{-\frac{1}{2}})$. Parameters are $q := e^{\hbar/\sqrt{\beta}} = e^{g_s R}$, $t := q^\beta = e^{\hbar\sqrt{\beta}} = e^{g_s \beta R}$, $p := q/t = e^{-\hbar(\sqrt{\beta}-1/\sqrt{\beta})}$, $u := t^\gamma$ and $v := (q/t)^{\frac{1}{2}}$. We will use the same letter p also for the set of power sums $p := (p_1, p_2, \cdots)$, but this appears only at $P_\lambda(x[\,p\,])$ or $Z_2(p)$.

2 Quantum deformation of $\mathcal{W}_N$ algebra

We start with recapitulating the results of the q-$\mathcal{W}_N$ algebra [20] [21] and define primary fields.

[1] The recursion relation for the 5D partition functions is derived recently in [38].

2.1 Bosons

We use three kinds of basis for bosons. First we define fundamental bosons h^i_n and Q^i_h for $i = 1, 2, \cdots, N$ and $n \in \mathbb{Z}$ such that

$$
\begin{aligned}
[h^i_n, h^j_m] &= \frac{1}{n}(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(t^{\frac{n}{2}} - t^{-\frac{n}{2}})\frac{[\,\delta_{ij}N - 1\,]_{p^n}}{[\,N\,]_{p^n}} p^{\frac{n}{2}N\mathrm{sgn}(j-i)}\delta_{n+m,0}, \\
[h^i_n, Q^j_h] &= \left(\delta_{ij} - \frac{1}{N}\right)\delta_{n,0}, \qquad [Q^i_h, Q^j_h] = 0, \qquad \sum_{i=1}^N p^{in} h^i_n = 0, \qquad \sum_{i=1}^N Q^i_h = 0 \quad (2.1)
\end{aligned}
$$

with q, $t := q^\beta \in \mathbb{C}$, $p := q/t$, $[\,n\,]_p := (p^{\frac{n}{2}} - p^{-\frac{n}{2}})/(p^{\frac{1}{2}} - p^{-\frac{1}{2}})$ and $\mathrm{sgn}(i) := 1$, 0 or -1 for $i > 0$, $i = 0$ or $i < 0$, respectively. Here $[A, B] := AB - BA$. This bosons correspond to the weights $\vec{h}_i$ of the vector representation whose inner product is $(\vec{h}_i \cdot \vec{h}_j) = \delta_{ij} - 1/N$. This algebra is invariant under the following involutions: $\omega^2_\pm = 1$,

$$
\begin{aligned}
\omega_+ : &\quad \sqrt{\beta} \mapsto 1/\sqrt{\beta}, \quad (q, t) \mapsto (t, q), \qquad h^i_n \mapsto h^{N-i+1}_n, \quad Q^i_h \mapsto Q^{N-i+1}_h, \quad (2.2) \\
\omega_- : &\quad \sqrt{\beta} \mapsto -\sqrt{\beta}, \quad (q, t) \mapsto (q^{-1}, t^{-1}), \quad h^i_n \mapsto h^{N-i+1}_n, \quad Q^i_h \mapsto Q^{N-i+1}_h. \quad (2.3)
\end{aligned}
$$

Next let us introduce root type bosons $\alpha^a_n := h^a_n - h^{a+1}_n$ and $Q^a_\alpha := Q^a_h - Q^{a+1}_h$ for $a = 1, 2, \cdots, N-1$. Then they satisfy

$$
\begin{aligned}
[\alpha^a_n, \alpha^b_m] &= \frac{1}{n}(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(t^{\frac{n}{2}} - t^{-\frac{n}{2}})C^{a,b}(p^n)\delta_{n+m,0}, \\
[\alpha^a_n, Q^b_\alpha] &= C^{a,b}(1)\delta_{n,0}, \qquad [Q^a_\alpha, Q^b_\alpha] = 0 \qquad (2.4)
\end{aligned}
$$

and

$$
\begin{aligned}
[h^i_n, \alpha^b_m] &= \frac{1}{n}(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(t^{\frac{n}{2}} - t^{-\frac{n}{2}})B^{i,b}(p^n)\delta_{n+m,0}, \\
[h^i_n, Q^b_\alpha] &= B^{i,b}(1)\delta_{n,0} = [\alpha^b_n, Q^i_h], \qquad [Q^i_h, Q^b_\alpha] = 0. \qquad (2.5)
\end{aligned}
$$

Here

$$
\begin{aligned}
B^{i,b}(p) &:= p^{\frac{1}{2}}\delta_{i,b} - p^{-\frac{1}{2}}\delta_{i-1,b}, \\
C^{a,b}(p) &:= -p^{-\frac{1}{2}}\delta_{a-1,b} + [\,2\,]_p\delta_{a,b} - p^{\frac{1}{2}}\delta_{a+1,b}. \qquad (2.6)
\end{aligned}
$$

Note that $[h^a_n + p^n h^{a+1}_n, \alpha^a_m] = 0$.

Finally we define weight type bosons $\Lambda^a_n := \sum_{b=1}^a h^b_n p^{(b-a-\frac{1}{2})n}$ and $Q^a_\Lambda := \sum_{b=1}^a Q^b_h$ for $a = 1, 2, \cdots, N-1$. Note that $h^a_n = p^{\frac{n}{2}}\Lambda^a_n - p^{-\frac{n}{2}}\Lambda^{a-1}_n$ with $\Lambda^0_n := 0$. Then they satisfy

$$
[\alpha^a_n, \Lambda^b_m] = \frac{1}{n}(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(t^{\frac{n}{2}} - t^{-\frac{n}{2}})\delta_{a,b}\delta_{n+m,0},
$$

$$[\alpha_n^a, Q_\Lambda^b] = \delta_{a,b}\delta_{n,0} = [\Lambda_n^b, Q_\alpha^a], \qquad [Q_\alpha^a, Q_\Lambda^b] = 0, \tag{2.7}$$

$$\begin{aligned}
[h_n^i, \Lambda_m^b] &= \frac{1}{n}(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(t^{\frac{n}{2}} - t^{-\frac{n}{2}})A^{i,b}(p^{-n})\delta_{n+m,0}, \\
[h_n^i, Q_\Lambda^b] &= A^{i,b}(1)\delta_{n,0} = [\Lambda_n^b, Q_h^i], \qquad [Q_h^i, Q_\Lambda^b] = 0
\end{aligned} \tag{2.8}$$

and

$$\begin{aligned}
[\Lambda_n^a, \Lambda_m^b] &= \frac{1}{n}(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(t^{\frac{n}{2}} - t^{-\frac{n}{2}})(C^{-1})^{a,b}(p^n)\delta_{n+m,0}, \\
[\Lambda_n^a, Q_\Lambda^b] &= (C^{-1})^{a,b}(1)\delta_{n,0}, \qquad [Q_\Lambda^a, Q_\Lambda^b] = 0.
\end{aligned} \tag{2.9}$$

Here

$$\begin{aligned}
A^{i,b}(p) &:= \frac{[\, N\theta(i \le b) - i\,]_p}{[\, N\,]_p} p^{\frac{1}{2}(b - N\theta(i>b))}, \\
(C^{-1})^{a,b}(p) &= \frac{[\, \min(a,b)\,]_p [\, N - \max(a,b)\,]_p}{[\, N\,]_p} p^{\frac{b-a}{2}}
\end{aligned} \tag{2.10}$$

with $\theta(P) := 1$ or 0 if the proposition P is true or false, respectively. Note that by $\omega_\pm$,

$$\begin{aligned}
\omega_\pm : \quad & \alpha_n^a \mapsto -\alpha_n^{N-a}, & Q_\alpha^a \mapsto -Q_\alpha^{N-a}, \\
\omega_\pm : \quad & \Lambda_n^a \mapsto -\Lambda_n^{N-a} p^{(a-N-\frac{1}{2})n}, & Q_\Lambda^a \mapsto -Q_\Lambda^{N-a}.
\end{aligned} \tag{2.11}$$

2.2 q-$\mathcal{W}_N$ algebra

Let us define fundamental vertices $\Lambda_i(z)$ and q-$\mathcal{W}_N$ generators $W^i(z)$ for $i = 1, 2, \cdots, N$ as follows:

$$\begin{aligned}
\Lambda_i(z) &:= :\exp\left\{\sum_{n\neq 0} h_n^i z^{-n}\right\}: q^{\sqrt{\beta}h_0^i} p^{\frac{N+1}{2}-i}, \\
W^i(zp^{\frac{1-i}{2}}) &:= \sum_{1\le j_1 < \cdots < j_i \le N} :\Lambda_{j_1}(z)\Lambda_{j_2}(zp^{-1})\cdots\Lambda_{j_i}(zp^{1-i}):
\end{aligned} \tag{2.12}$$

and $W^0(z) := 1$. Here $:*:$ stands for the usual bosonic normal ordering such that the bosons h_n^i with non-negative mode $n \ge 0$ are in the right. Note that

$$W^N(zp^{\frac{1-N}{2}}) = :\Lambda_1(z)\Lambda_2(zp^{-1})\cdots\Lambda_N(zp^{1-N}): = 1. \tag{2.13}$$

These generators are obtained from the following quantum Miura transformation:

$$\sum_{i=0}^{N}(-1)^i W^i(zp^{\frac{1-i}{2}})p^{(N-i)D_z} = :\left(p^{D_z} - \Lambda_1(z)\right)\left(p^{D_z} - \Lambda_2(zp^{-1})\right)\cdots\left(p^{D_z} - \Lambda_N(zp^{1-N})\right): \tag{2.14}$$

with $D_z := z\frac{\partial}{\partial z}$. Remark that p^{D_z} is the p-shift operator such that $p^{D_z}f(z) = f(pz)$. The mode n generator W_n^i is defined by $W^i(z) =: \sum_{n\in\mathbb{Z}} W_n^i z^{-n}$.

2.3 Screening currents

By using root type bosons we define screening currents $S_\pm^a(z)$ as follows:

$$S_\pm^a(z) := :\exp\left\{\mp\sum_{n\neq 0}\frac{\alpha_n^a}{\xi_\pm^{\frac{n}{2}}-\xi_\pm^{-\frac{n}{2}}}z^{-n}\right\}: e^{\pm\sqrt{\beta}^{\pm 1}Q_\alpha^a}z^{\pm\sqrt{\beta}^{\pm 1}\alpha_0^a}, \qquad \xi_+=q, \quad \xi_-=t. \tag{2.15}$$

Note that the Langlands duality $\omega_-\omega_+S_+^a(z)=S_-^a(z)$. We denote the negative mode part of $S_\pm^a(z)$ by $(S_\pm^a(z))_- := \exp\left\{\mp\sum_{n<0}\frac{\alpha_n^a}{\xi_\pm^{\frac{n}{2}}-\xi_\pm^{-\frac{n}{2}}}z^{-n}\right\}$. The screening currents satisfy

$$\begin{aligned}&[:\left(p^{D_z}-\Lambda_1(z)\right)\left(p^{D_z}-\Lambda_2(zp^{-1})\right)\cdots\left(p^{D_z}-\Lambda_N(zp^{1-N})\right):, S_\pm^a(w)]\\ &\quad=(q^{\frac12}-q^{-\frac12})(t^{\frac12}-t^{-\frac12})\frac{d}{d_{\xi_\pm}w}:\left(p^{D_z}-\Lambda_1(z)\right)\cdots\left(p^{D_z}-\Lambda_{a-1}(zp^{2-a})\right)\\ &\qquad\times w\delta\left(\frac{w}{z}p^{a-1}\right)A_\pm^a(w)p^{D_z}\left(p^{D_z}-\Lambda_{a+2}(zp^{-1-a})\right)\cdots\left(p^{D_z}-\Lambda_N(zp^{1-N})\right):\end{aligned} \tag{2.16}$$

with

$$A_\pm^a(w) := :\exp\left\{\sum_{n\neq 0}\frac{\xi_\pm^{\pm\frac{n}{2}}h_n^{a+1}-\xi_\pm^{\mp\frac{n}{2}}h_n^a}{\xi_\pm^{\pm\frac{n}{2}}-\xi_\pm^{\mp\frac{n}{2}}}w^{-n}\right\}: e^{\pm\sqrt{\beta}^{\pm 1}Q_\alpha^a}w^{\pm\sqrt{\beta}^{\pm 1}\alpha_0^a}\xi_\pm^{\frac12\sqrt{\beta}^{\pm 1}(h_0^a+h_0^{a+1})}p^{\frac{N}{2}-a} \tag{2.17}$$

and $\frac{d}{d_\xi w}f(w) := (f(\xi^{\frac12}w)-f(\xi^{-\frac12}w))/((\xi^{\frac12}-\xi^{-\frac12})w)$. Here we use the identity

$$\exp\left\{\sum_{n>0}\frac{1-t^n}{n}x^n\right\}-t\exp\left\{\sum_{n>0}\frac{1-t^{-n}}{n}x^{-n}\right\}=(1-t)\delta(x) \tag{2.18}$$

with the multiplicative δ-function $\delta(z) := \sum_{n\in\mathbb{Z}}z^n$ satisfying $\delta(z)f(z)=\delta(z)f(1)$. Therefore the screening currents $S_\pm^a(z)$ commute with any q-$\mathcal{W}_N$ generators up to total difference. Thus screening charges $\oint dz S_\pm^a(z)$ commute with any q-$\mathcal{W}_N$ generators

$$[\oint dz S_\pm^a(z), W^b(w)]=0, \qquad a,b=1,2,\cdots,N-1. \tag{2.19}$$

For a Laurent series $f(z) := \sum_{n\in\mathbb{Z}}f_nz^n$ in z, the integral $\oint\frac{dz}{2\pi iz}f(z)$ stands for the constant term in $f(z)$, i.e.,

$$\oint\frac{dz}{2\pi iz}\sum_{n\in\mathbb{Z}}f_nz^n := \mathrm{CT}_{\{z\}}\sum_{n\in\mathbb{Z}}f_nz^n := f_0. \tag{2.20}$$

If f is multivalued function, we should choose an appropriate cycle or need to introduce a pseudo-constant to make it single-valued one. (see (2.32)).[2]

[2] One can replace the integral $\oint dz$ by any linear map which satisfies $\oint dz\frac{d}{d_\xi z}f(z)=0$ with $\xi=\xi_\pm$, for example, by the Jackson integral, provided $f(0)=f(1)$.

2.4 Primary fields and degenerate operators

For parameters u and γ with $u := t^\gamma$, let us define the following vertex operators

$$V_u^a(z) := \,:\!\exp\left\{\sum_{n\neq 0}\frac{(u^{\frac{n}{2}}-u^{-\frac{n}{2}})\Lambda_n^a}{(q^{\frac{n}{2}}-q^{-\frac{n}{2}})(t^{\frac{n}{2}}-t^{-\frac{n}{2}})}z^{-n}\right\}\!:\, e^{-\gamma\sqrt{\beta}Q_\Lambda^a}z^{-\gamma\sqrt{\beta}\Lambda_0^a}. \tag{2.21}$$

They satisfy

$$g_{u,p}^{a,L}\left(\frac{w}{z}\right)\Lambda_i(z)V_u^a(w)-V_u^a(w)\Lambda_i(z)g_{u,p}^{a,R}\left(\frac{z}{w}\right)=(u^{-1}-1)\sum_{b=1}^{a}\delta_{i,b}\delta\left(\frac{w}{zu^{\frac{1}{2}}}\right):\!\Lambda_i(z)V_u^a(w)\!:, \tag{2.22}$$

where $g_{u,p}^{a,L}(x)$ and $g_{u,p}^{a,R}(x)$ are inverse of the OPE factors,

$$\begin{aligned}
g_{u,p}^{a,L}(x) &:= \frac{:\!\Lambda_j(z)V_u^a(w)\!:}{\Lambda_j(z)V_u^a(w)} = \exp\left\{\sum_{n>0}\frac{u^{-\frac{n}{2}}-u^{\frac{n}{2}}}{n}\frac{[\,a\,]_{p^n}}{[\,N\,]_{p^n}}p^{\frac{n}{2}(a-N)}x^n\right\}u^{-\frac{a}{N}},\\
g_{u,p}^{a,R}(x) &:= \frac{:\!V_u^a(w)\Lambda_j(z)\!:}{V_u^a(w)\Lambda_j(z)} = \exp\left\{\sum_{n>0}\frac{u^{\frac{n}{2}}-u^{-\frac{n}{2}}}{n}\frac{[\,a\,]_{p^n}}{[\,N\,]_{p^n}}p^{\frac{n}{2}(N-a)}x^n\right\}
\end{aligned} \tag{2.23}$$

for any $j > a$. Since p^{D_z} commutes with $V_u^a(w)$,

$$\begin{aligned}
&\left(p^{D_z}-g_{u,p}^{a,L}\left(\frac{w}{z}\right)\Lambda_i(z)\right)V_u^a(w)-V_u^a(w)\left(p^{D_z}-\Lambda_i(z)g_{u,p}^{a,R}\left(\frac{z}{w}\right)\right)\\
&\qquad=(1-u^{-1})\sum_{b=1}^{a}\delta_{i,b}\delta\left(\frac{w}{zu^{\frac{1}{2}}}\right):\!\Lambda_i(z)V_u^a(w)\!:.
\end{aligned} \tag{2.24}$$

Remark that $\Lambda_{N-a+1}(z)$ and $V_u^{N-a}(w)$ satisfy same relation with replacing $p\leftrightarrow p^{-1}$ and $u\leftrightarrow u^{-1}$. By using (2.24) and ω_-, we have

Proposition. *The vertex operators $V_u^1(w)$ and $V_u^{N-1}(w)$ enjoy the following relations:*

$$\begin{aligned}
&:\!\left(p^{D_z}-g_{u,p}^{1,L}\left(\frac{w}{z}\right)\Lambda_1(z)\right)\cdots\left(p^{D_z}-g_{u,p}^{1,L}\left(\frac{w}{zp^{1-N}}\right)\Lambda_N(zp^{1-N})\right)\!:V_u^1(w)\\
&\quad-V_u^1(w):\!\left(p^{D_z}-\Lambda_1(z)g_{u,p}^{1,R}\left(\frac{z}{w}\right)\right)\cdots\left(p^{D_z}-\Lambda_N(zp^{1-N})g_{u,p}^{1,R}\left(\frac{zp^{1-N}}{w}\right)\right)\!:\\
&\qquad=(1-u^{-1})\delta\left(\frac{w}{zu^{\frac{1}{2}}}\right):\!\Lambda_1(z)V_u^1(w)\left(p^{D_z}-\Lambda_2(zp^{-1})\right)\cdots\left(p^{D_z}-\Lambda_N(zp^{1-N})\right)\!:,
\end{aligned} \tag{2.25}$$

$$\begin{aligned}
&:\!\left(p^{D_z}-g_{1/u,1/p}^{1,L}\left(\frac{w}{z}\right)\Lambda_1(z)\right)\cdots\left(p^{D_z}-g_{1/u,1/p}^{1,L}\left(\frac{w}{zp^{1-N}}\right)\Lambda_N(zp^{1-N})\right)\!:V_u^{N-1}(w)\\
&\quad-V_u^{N-1}(w):\!\left(p^{D_z}-\Lambda_1(z)g_{1/u,1/p}^{1,R}\left(\frac{z}{w}\right)\right)\cdots\left(p^{D_z}-\Lambda_N(zp^{1-N})g_{1/u,1/p}^{1,R}\left(\frac{zp^{1-N}}{w}\right)\right)\!:
\end{aligned}$$

$$= (1-u)\delta\left(\frac{wu^{\frac{1}{2}}}{zp^{1-N}}\right) \mathbf{:} \left(p^{D_z} - \Lambda_1(z)\right)\cdots\left(p^{D_z} - \Lambda_{N-1}(zp^{2-N})\right)\Lambda_N(zp^{1-N})V_u^{N-1}(w)\mathbf{:}.$$

Expanding (2.25) gives the relation with the q-$\mathcal{W}_N$ generators $W^i(z)$.

When $u = t$ or q^{-1}, let $V_+^a(z) := V_t^a(z)$ and $V_-^a(z) := V_{q^{-1}}^a(z)$, i.e.,

$$V_\pm^a(z) := \mathbf{:}\exp\left\{\pm\sum_{n\neq 0}\frac{\Lambda_n^a}{\xi_\pm^{\frac{n}{2}} - \xi_\pm^{-\frac{n}{2}}} z^{-n}\right\}\mathbf{:}\, e^{\mp\sqrt{\beta}^{\pm 1} Q_\Lambda^a} z^{\mp\sqrt{\beta}^{\pm 1}\Lambda_0^a} \tag{2.26}$$

with $\xi_+ := q$, $\xi_- := t$. As a generalization of the the $(\ell+1, k+1)$ operators in the $N = 2$ case [23], we can define a q-deformation of the degenerate operators for $\ell, k \in \mathbb{Z}_{\geq 0}$,

$$\begin{aligned} V_{\ell+1,k+1}^a(z) &:= \mathbf{:}\prod_{i=1}^{\ell} V_+^a(q^{\frac{\ell+1-2i}{2\ell}} z)\prod_{j=1}^{k} V_-^a(t^{\frac{k+1-2j}{2k}} z)\mathbf{:} \\ &= \mathbf{:}\exp\left\{\sum_{n\neq 0}\left(\frac{1}{q^{\frac{n}{2\ell}} - q^{-\frac{n}{2\ell}}} - \frac{1}{t^{\frac{n}{2k}} - t^{-\frac{n}{2k}}}\right)\Lambda_n^a z^{-n}\right\}\mathbf{:}\, e^{\alpha Q_\Lambda^a} z^{\alpha\Lambda_0^a} \end{aligned} \tag{2.27}$$

with $\alpha := -\ell\sqrt{\beta} + k/\sqrt{\beta}$ and $1/(q^{\frac{n}{2\ell}} - q^{-\frac{n}{2\ell}})|_{\ell=0} := 0$.

2.5 Boson Fock space

Next we refer to the representation of the q-$\mathcal{W}_N$ algebra. Let $\mathcal{F}_\alpha$ be the boson Fock space generated by the highest weight state $|\alpha\rangle$ such that $\alpha_n^a|0\rangle = 0$ for $n \geq 0$ and $|\alpha\rangle := \exp\{\sum_{a=1}^{N-1}\alpha^a Q_\Lambda^a\}|0\rangle$. Note that $\alpha_0^a|\alpha\rangle = \alpha^a|\alpha\rangle$. The dual module $\mathcal{F}_\alpha^*$ is generated by $\langle\alpha|$ such that $\langle 0|\alpha_{-n}^a = 0$ for $n \geq 0$ and $\langle\alpha| := \langle 0|\exp\{-\sum_{a=1}^{N-1}\alpha^a Q_\Lambda^a\}$. The bilinear form $\mathcal{F}_\alpha^* \otimes \mathcal{F}_\alpha \to \mathbb{C}$ is uniquely defined by $\langle 0|0\rangle = 1$.

2.6 Highest weight module of q-$\mathcal{W}_N$ algebra

Let $|\lambda\rangle$ be the highest weight vector of the q-$\mathcal{W}_N$ algebra which satisfies $W_n^a|\lambda\rangle = 0$ for $n > 0$ and $a = 1, 2, \cdots, N-1$ and $W_0^a|\lambda\rangle = \lambda^a|\lambda\rangle$ with $\lambda^a \in \mathbb{C}$. Let M_λ be the Verma module over the q-$\mathcal{W}_N$ algebra generated by $|\lambda\rangle$. The dual module M_λ^* is generated by $\langle\lambda|$ such that $\langle\lambda|W_n^a = 0$ for $n < 0$ and $\langle\lambda|W_0^a = \lambda^a\langle\lambda|$. The bilinear form $M_\lambda^* \otimes M_\lambda \to \mathbb{C}$ is uniquely defined by $\langle\lambda|\lambda\rangle = 1$. A singular vector $|\chi\rangle \in M_\lambda$ is defined by $W_n^a|\chi\rangle = 0$ for $n > 0$ and $W_0^a|\chi\rangle = (\lambda^a + N^a)|\chi\rangle$ with $N^a \in \mathbb{C}$.

The highest weight vector $|\alpha\rangle \in \mathcal{F}_\alpha$ of the boson algebra is also that of the q-$\mathcal{W}_N$ algebra, i.e., $W_n^a|\alpha\rangle = 0$ for $n > 0$ and $a = 1, 2, \cdots, N-1$. Note that $W_0^a|0\rangle = [N]_p^a|0\rangle$ with $[N]_p := (p^{\frac{N}{2}} - p^{-\frac{N}{2}})/(p^{\frac{1}{2}} - p^{-\frac{1}{2}})$.

2.7 Singular vectors

For a set of non-negative integers s_a and $r_a \geq r_{a+1} \geq 0$ with $a = 1, \cdots, N-1$, let

$$\begin{aligned} \pm\, \alpha^{\pm,a}_{r,s} &:= (1 + r_a - r_{a-1})\sqrt{\beta}^{\pm 1} - (1 + s_a)\sqrt{\beta}^{\mp 1}, \qquad r_0 := 0, \\ \pm \widetilde{\alpha}^{\pm,a}_{r,s} &:= (1 - r_a + r_{a+1})\sqrt{\beta}^{\pm 1} - (1 + s_a)\sqrt{\beta}^{\mp 1}, \qquad r_N := 0. \end{aligned} \tag{2.28}$$

The singular vectors $|\chi^{\pm}_{rs}\rangle \in \mathcal{F}_{\alpha^{\pm}_{rs}}$ are realized by the screening currents as follows:

$$\begin{aligned} |\chi^{\pm}_{r,s}\rangle &= \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r_a} \frac{dz^a_j}{2\pi i} \cdot S^1_{\pm}(z^1_1) \cdots S^1_{\pm}(z^1_{r_1}) \cdots S^{N-1}_{\pm}(z^{N-1}_1) \cdots S^{N-1}_{\pm}(z^{N-1}_{r_{N-1}}) |\widetilde{\alpha}^{\pm}_{r,s}\rangle \\ &= \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r_a} \frac{dz^a_j}{2\pi i z^a_j} (z^a_j)^{-s_a} (S^a_{\pm}(z^a_j))_- \cdot \Delta^{qW}(z^a; \xi_{\pm}, \xi_{\mp}) \Pi\left(\overline{z^a}, p z^{a+1}; \xi_{\pm}, \xi_{\mp}\right) |\alpha^{\pm}_{r,s}\rangle \end{aligned} \tag{2.29}$$

with $z^N := 0$, $\overline{z} := 1/z$, $\xi_+ := q$ and $\xi_- := t$. Note that $\omega_- \omega_+ |\chi^+_{r,s}\rangle = |\chi^-_{r,s}\rangle$. Here

$$\begin{aligned} \Pi(z, w) := \Pi(z, w; q, t) &:= \prod_{i,j} \exp\left\{ \sum_{n>0} \frac{1}{n} \frac{t^{\frac{n}{2}} - t^{-\frac{n}{2}}}{q^{\frac{n}{2}} - q^{-\frac{n}{2}}} p^{-\frac{n}{2}} z^n_i w^n_j \right\} \\ &= \prod_{i,j} \prod_{\ell \geq 0} \frac{1 - q^{\ell} t z_i w_j}{1 - q^{\ell} z_i w_j}, \qquad |q| < 1, \end{aligned} \tag{2.30}$$

$$\begin{aligned} \Delta^{qW}(z) := \Delta^{qW}(z; q, t) &:= \prod_{i<j} \exp\left\{ -\sum_{n>0} \frac{1}{n} \frac{t^{\frac{n}{2}} - t^{-\frac{n}{2}}}{q^{\frac{n}{2}} - q^{-\frac{n}{2}}} (p^{\frac{n}{2}} + p^{-\frac{n}{2}}) \frac{z^n_j}{z^n_i} \right\} \cdot \prod_{i=1}^{r} z_i^{(r+1-2i)\beta} \\ &= \prod_{i<j} (1 - z_j/z_i) \prod_{\ell \geq 0} \frac{1 - q^{\ell} p z_j/z_i}{1 - q^{\ell} t z_j/z_i} \cdot \prod_{i=1}^{r} z_i^{(r+1-2i)\beta}, \quad |q| < 1 \end{aligned} \tag{2.31}$$

with $\beta := \log t / \log q$. Note that $\Delta^{qW}(cz) = \Delta^{qW}(z)$.

When $\beta \notin \mathbb{Z}$, $\Delta^{qW}(z)$ becomes a multivalued function but we can replace it with the following single valued one

$$\widetilde{\Delta^{qW}}(z) := \Delta^{qW}(z) \prod_{i<j} F(z_j/z_i) \tag{2.32}$$

with the pseudo-constant $F(x) = F(qx)$ defined by

$$F(x) := x^{\beta - 1} \frac{\vartheta_q(tx)}{\vartheta_q(qx)}. \tag{2.33}$$

Here

$$\vartheta_q(x) := \prod_{\ell \geq 0} (1 - q^{\ell} x)(1 - q^{\ell+1}/x)(1 - q^{\ell+1}) \tag{2.34}$$

is the ϑ-function with the multiplicative period q.

3 Quantum deformation of β-ensemble

Note that the singular vector in (2.29) is naturally mapped to the Macdonald polynomial [24] defined in the appendix A [25][21]. As a generalization of this map one can define, under the strategy of [3], a quantum deformation of the generalized matrix model, i.e., q-deformed β-ensemble.

3.1 Isomorphisms between bosons

With a new parameters $p^{(a)} := (p_1^{(a)}, p_2^{(a)}, \cdots)$ let us define the following vertex operator

$$V_N := \prod_{a=1}^{N-1} \exp\left\{\sum_{n>0} \frac{\Lambda_n^a}{q^{\frac{n}{2}} - q^{-\frac{n}{2}}} p_n^{(a)}\right\}. \tag{3.1}$$

Note that $[\Lambda_n^a, \Lambda_m^b] = 0$ for $n, m > 0$. Then $\langle\alpha|V_N$ defines the isomorphism between the boson algebras $\langle h_n^a\rangle_{n\in\mathbb{Z}}^{1\le a<N}$ and $\langle p_n^{(a)}, \alpha^a, \frac{\partial}{\partial p_n^{(a)}}\rangle_{n\in\mathbb{N}}^{1\le a<N}$ by

$$\begin{aligned}\frac{t^{\frac{n}{2}} - t^{-\frac{n}{2}}}{n} p_n^{(a)} \langle\alpha|V_N &= \langle\alpha|V_N \alpha_{-n}^a,\\ (q^{\frac{n}{2}} - q^{-\frac{n}{2}}) \frac{\partial}{\partial p_n^{(a)}} \langle\alpha|V_N &= \langle\alpha|V_N \Lambda_n^a\end{aligned} \tag{3.2}$$

for $n > 0$ and $\alpha^a\langle\alpha|V_N = \langle\alpha|V_N\alpha_0^a$. Since $h_n^i = \sum_{b=1}^{N-1} A^{i,b}(p^n)\alpha_n^b = \sum_{b=1}^{N-1} B^{i,b}(p^n)\Lambda_n^b$, we have

$$\begin{aligned}\frac{t^{\frac{n}{2}} - t^{-\frac{n}{2}}}{n} \sum_{b=1}^{N-1} A^{i,b}(p^{-n}) p_n^{(b)} \langle\alpha|V_N &= \langle\alpha|V_N h_{-n}^i,\\ (q^{\frac{n}{2}} - q^{-\frac{n}{2}}) \sum_{b=1}^{N-1} B^{i,b}(p^n) \frac{\partial}{\partial p_n^{(b)}} \langle\alpha|V_N &= \langle\alpha|V_N h_n^i\end{aligned} \tag{3.3}$$

for $n > 0$ and $h^i\langle\alpha|V_N = \langle\alpha|V_N h_0^i$ with $h^i = \left[\sum_{b=i}^{N-1} - \sum_{b=1}^{N-1} b/N\right]\alpha^b$.

The vector $|S_{r,s}^+\rangle := \prod_{a=1}^{N-1}\prod_{k=1}^{r_a}(S_+^a(z_k^a))_- \cdot |\alpha_{r,s}^+\rangle$ in (2.29) also defines another linear map from $\langle h_n^a\rangle_{n\in\mathbb{N}}^{1\le a<N}$ to $\langle\sum_{k=1}^{r_a}(z_k^a)^n\rangle^{1\le a<N}$ by

$$\Lambda_n^a|S_{r,s}^+\rangle = |S_{r,s}^+\rangle \frac{t^{\frac{n}{2}} - t^{-\frac{n}{2}}}{n} \sum_{k=1}^{r_a} (z_k^a)^n, \qquad n > 0. \tag{3.4}$$

Then

$$h_n^i|S_{r,s}^+\rangle = |S_{r,s}^+\rangle \frac{t^{\frac{n}{2}} - t^{-\frac{n}{2}}}{n} \sum_{b=1}^{N-1} B^{i,b}(p^n) \sum_{k=1}^{r_b} (z_k^b)^n, \qquad n > 0. \tag{3.5}$$

3.2 q-deformed β-ensemble

Let us define the following partition function

Definition. *Let* $Z_N := Z_N\left(\{p^{(a)}\}_{a=1}^{N-1}\right) := \langle \alpha_{r,s}^+|V_N|\chi_{r,s}^+\rangle$.

Then by (2.29), (2.15) and (3.2), we have

$$\begin{aligned} Z_N &= \oint \prod_{a=1}^{N-1}\prod_{j=1}^{r_a} \frac{dz_j^a}{2\pi i}\langle \alpha_{r,s}^+|V_N S_+^1(z_1^1)\cdots S_+^1(z_{r_1}^1)\cdots S_+^{N-1}(z_1^{N-1})\cdots S_+^{N-1}(z_{r_{N-1}}^{N-1})|\widetilde{\alpha}_{r,s}^+\rangle \\ &= \oint \prod_{a=1}^{N-1}\prod_{j=1}^{r_a} \frac{dz_j^a}{2\pi i z_j^a}(z_j^a)^{-s_a}\exp\left\{\sum_{n>0}\frac{1}{n}\frac{t^{\frac{n}{2}}-t^{-\frac{n}{2}}}{q^{\frac{n}{2}}-q^{-\frac{n}{2}}}(z_j^a)^n p_n^{(a)}\right\}\cdot \Delta^{qW}(z^a)\Pi\left(\overline{z^a}, pz^{a+1}\right) \\ &= \oint \prod_{a=1}^{N-1}\prod_{j=1}^{r_a}\frac{dz_j^a}{2\pi i}\cdot\prod_{a=1}^{N-1}\Delta^{qW}(z^a)e^{W(z^a,z^{a+1})}, \end{aligned} \tag{3.6}$$

$$W(z^a,z^{a+1}) := \sum_{n>0}\frac{1}{n}\frac{t^{\frac{n}{2}}-t^{-\frac{n}{2}}}{q^{\frac{n}{2}}-q^{-\frac{n}{2}}}\sum_{i=1}^{r_a}\left\{(z_i^a)^n p_n^{(a)} + \sum_{j=1}^{r_{a+1}}\left(\frac{p^{\frac{1}{2}}z_j^{a+1}}{z_i^a}\right)^n\right\} - (s_a+1)\sum_{i=1}^{r_a}\log z_i^a. \tag{3.7}$$

Here $z^N := 0$. This Z_N is regarded as a q-deformation of the partition function of the generalized matrix model [3], i.e., β-ensemble. One can define other type of partition functions by acting involutions (2.2), (2.3) and (A.10).

We can calculate this integral by using the Macdonald polynomials $P_\lambda(x)$ with the Young diagram λ, their fusion coefficient $f_{\lambda,\mu}^{\nu}$ and the inner products $\langle *,*\rangle$, $\langle *,*\rangle_r'$ and $\langle *,*\rangle_r''$ defined in the appendix A. By the Cauchy formula (A.9), the Galilean boost (A.18) and (A.12) we have

$$\begin{aligned} Z_N &= \oint \prod_{a=1}^{N-1}\prod_{j=1}^{r_a}\frac{dz_j^a}{2\pi i z_j^a}\cdot P_{(s_a^{r_a})}(\bar{z}^a)\sum_{\lambda_a}\frac{P_{\lambda_a}(z^a)P_{\lambda_a}(x[p^a])}{\langle\lambda_a\rangle}\Delta^{qW}(z^a)\sum_{\mu_a}\frac{P_{\mu_a}(\bar{z}^a)P_{\mu_a}(pz^{a+1})}{\langle\mu_a\rangle} \\ &= \oint \prod_{a=1}^{N-1}\prod_{j=1}^{r_a}\frac{dz_j^a}{2\pi i z_j^a}\cdot\sum_{\lambda_a,\mu_a,\nu_a} f_{\mu_{a-1},\lambda_a}^{\nu_a}P_{\nu_a}(z^a)\frac{P_{\lambda_a}(x[p^a])}{\langle\lambda_a\rangle}\Delta^{qW}(z^a)p^{|\mu_a|}\frac{P_{\mu_a+(s_a^{r_a})}(\bar{z}^a)}{\langle\mu_a\rangle} \end{aligned} \tag{3.8}$$

with $\mu_0 = \mu_{N-1} := (0)$. Here λ_a, μ_a and ν_a are Young diagrams such that $\lambda_{a,i}\geq\lambda_{a,i+1}$, and so on. $P_\lambda(x[p])$ denotes the Macdonald function in power sums $p := (p_1, p_2, \cdots)$. By the orthogonality with respect to the inner product $\langle *,*\rangle_r''$ in (A.7), we obtain

Proposition.

$$Z_N = \prod_{a=1}^{N-1}\sum_{\lambda_a,\mu_a} f_{\mu_{a-1},\lambda_a}^{\mu_a+(s_a^{r_a})}P_{\mu_a+(s_a^{r_a})}(z^a)\frac{P_{\lambda_a}(x[p^a])}{\langle\lambda_a\rangle}p^{|\mu_a|}\frac{r_a!\langle\mu_a+(s_a^{r_a})\rangle_{r_a}''}{\langle\mu_a\rangle} \tag{3.9}$$

with $\langle 0\rangle := 1$.

For any λ with $\lambda_1 \leq s$, let $\widehat{\lambda}$ be its complements with respect to (s^r), i.e., $\widehat{\lambda}_i = s - \lambda_{r-i+1}$. The fusion coefficient $f^{\nu}_{\lambda,\mu}$ defined in (A.12) satisfies $f^{\nu}_{\lambda,(0)} = \delta_{\lambda,\nu}$ and $f^{(s^r)}_{\lambda,\mu} = \delta_{\mu,\widehat{\lambda}} f^{(s^r)}_{\lambda,\widehat{\lambda}}$. Since $\mu_0 = \mu_{N-1} = (0)$, we have $\lambda_1 = \mu_1 + (s_1^{r_1})$ and $\lambda_{N-1} = \widehat{\mu}_{N-2}$ with respect to $(s_{N-1}^{r_{N-1}})$. Therefore (3.9) is summed over $(N-2)+(N-3)$ Young diagrams for $N \geq 3$.

For any function $\mathcal{O}$ in z_j^a's, the correlation function with respect to $\mathcal{O}$ is defined by

$$\langle\!\langle \mathcal{O} \rangle\!\rangle := \frac{1}{Z_N} \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r_a} \frac{dz_j^a}{2\pi i} \cdot \mathcal{O} \prod_{a=1}^{N-1} \Delta^{qW}(z^a) e^{W(z^a, z^{a+1})}. \tag{3.10}$$

The effective action S_{eff} defined by $Z_N =: \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r_a} \frac{dz_j^a}{2\pi i} \cdot e^{S_{\text{eff}}}$ is now

$$S_{\text{eff}} = \sum_{a=1}^{N-1} W(z^a, z^{a+1}) - \sum_{n>0} \frac{[\,2\,]_{p^n}}{n} \frac{t^{\frac{n}{2}} - t^{-\frac{n}{2}}}{q^{\frac{n}{2}} - q^{-\frac{n}{2}}} \sum_{a=1}^{N-1} \sum_{i<j} \left(\frac{z_j^a}{z_i^a}\right)^n + \beta \sum_{a=1}^{N-1} \sum_{i=1}^{r_a} (r_a + 1 - 2i) \log z_i^a. \tag{3.11}$$

The saddle point condition is $\frac{\partial S_{\text{eff}}}{\partial z_k^a} = 0$ with

$$\begin{aligned} z_k^a \frac{\partial S_{\text{eff}}}{\partial z_k^a} = & \sum_{n>0} \frac{t^{\frac{n}{2}} - t^{-\frac{n}{2}}}{q^{\frac{n}{2}} - q^{-\frac{n}{2}}} \left\{ (z_k^a)^n p_n^{(a)} - [\,2\,]_{p^n} \left(\sum_{i<k} \left(\frac{z_k^a}{z_i^a}\right)^n - \sum_{j>k} \left(\frac{z_j^a}{z_k^a}\right)^n \right) \right. \\ & \left. + \sum_{i=1}^{r_{a-1}} \left(\frac{p^{\frac{1}{2}} z_k^a}{z_i^{a-1}} \right)^n - \sum_{j=1}^{r_{a+1}} \left(\frac{p^{\frac{1}{2}} z_j^{a+1}}{z_k^a} \right)^n \right\} + ((r_a + 1 - 2k)\beta - s_a - 1). \end{aligned} \tag{3.12}$$

3.3 q-$\mathcal{W}_N$ constraint

Next let us define $\hat{\Lambda}_i(z)$ and $\mathcal{W}^i(z)$ as follows, which are the power sum realization of fundamental vertices $\Lambda_i(z)$ and q-$\mathcal{W}_N$ generators $W^i(z)$, respectively:

$$\begin{aligned} \hat{\Lambda}_i(z) := & \exp\left\{ \sum_{n>0} \frac{t^{\frac{n}{2}} - t^{-\frac{n}{2}}}{n} z^n \sum_{b=1}^{N-1} A^{i,b}(p^{-n}) p_n^{(b)} \right\} \\ & \times \exp\left\{ \sum_{n>0} (q^{\frac{n}{2}} - q^{-\frac{n}{2}}) z^{-n} \sum_{b=1}^{N-1} B^{i,b}(p^n) \frac{\partial}{\partial p_n^{(b)}} \right\} q^{\sqrt{\beta} h^i_{r,s}} p^{\frac{N+1}{2} - i}, \end{aligned} \tag{3.13}$$

$$\sum_{i=0}^{N} (-1)^i \mathcal{W}^i(z p^{\frac{1-i}{2}}) p^{(N-i)D_z} := \mathbf{:} \left(p^{D_z} - \hat{\Lambda}_1(z)\right) \left(p^{D_z} - \hat{\Lambda}_2(zp^{-1})\right) \cdots \left(p^{D_z} - \hat{\Lambda}_N(zp^{1-N})\right) \mathbf{:} \tag{3.14}$$

and $\mathcal{W}^i(z) =: \sum_{n\in\mathbb{Z}} \mathcal{W}^i_n z^{-n}$. Here $\mathbf{:} * \mathbf{:}$ stands for the normal ordering such that the differential operators $\frac{\partial}{\partial p_n^{(a)}}$ are in the right. Then by the isomorphism (3.3),

$$\hat{\Lambda}_i(z) \langle \alpha_{r,s} | V_N = \langle \alpha_{r,s} | V_N \Lambda_i(z), \qquad \mathcal{W}^i(z) \langle \alpha_{r,s} | V_N = \langle \alpha_{r,s} | V_N W^i(z). \tag{3.15}$$

Therefore the highest weight condition for the singular vector $W_n^a|\chi\rangle = 0$ for $n > 0$ is equivalent to the following q-$\mathcal{W}_N$ constraint:

Theorem.

$$\mathcal{W}_n^a Z_N = 0, \qquad n > 0. \tag{3.16}$$

3.4 Loop equation and quantum spectral curve

Let us define $\widetilde{\Lambda}_i(z)$ and $\widetilde{\mathcal{W}}^i(z)$ as follows, which correspond to fundamental vertices $\Lambda_i(z)$ and q-$\mathcal{W}_N$ generators $W^i(z)$, respectively:

$$\begin{aligned}\widetilde{\Lambda}_i(z) :=& \exp\left\{\sum_{n>0}\frac{t^{\frac{n}{2}}-t^{-\frac{n}{2}}}{n}z^n\sum_{b=1}^{N-1}A^{i,b}(p^{-n})p_n^{(b)}\right\}\\ &\times\exp\left\{\sum_{n>0}\frac{t^{\frac{n}{2}}-t^{-\frac{n}{2}}}{n}z^{-n}\sum_{b=1}^{N-1}B^{i,b}(p^n)\sum_{k=1}^{r_b}(z_k^b)^n\right\}q^{\sqrt{\beta}h_{r,s}^i}p^{\frac{N+1}{2}-i},\end{aligned} \tag{3.17}$$

$$\sum_{i=0}^{N}(-1)^i\widetilde{\mathcal{W}}^i(zp^{\frac{1-i}{2}})p^{(N-i)D_z} := \left(p^{D_z}-\widetilde{\Lambda}_1(z)\right)\left(p^{D_z}-\widetilde{\Lambda}_2(zp^{-1})\right)\cdots\left(p^{D_z}-\widetilde{\Lambda}_N(zp^{1-N})\right) \tag{3.18}$$

and $\widetilde{\mathcal{W}}^i(z) =: \sum_{n\in\mathbb{Z}}\widetilde{\mathcal{W}}_n^i z^{-n}$. Then by linear maps (3.3) and (3.4), we have

$$\begin{aligned}\langle\alpha_{r,s}^+|V_N\Lambda_i(z)|S_{r,s}^+\rangle &= \langle\alpha_{r,s}^+|V_N|S_{r,s}^+\rangle\widetilde{\Lambda}_i(z),\\ \langle\alpha_{r,s}^+|V_N W^i(z)|S_{r,s}^+\rangle &= \langle\alpha_{r,s}^+|V_N|S_{r,s}^+\rangle\widetilde{\mathcal{W}}^i(z).\end{aligned} \tag{3.19}$$

Hence

$$\frac{1}{Z_N}\langle\alpha_{r,s}^+|V_N W^i(z)|\chi_{r,s}^+\rangle = \left\langle\!\!\left\langle\, \widetilde{\mathcal{W}}^i(z)\,\right\rangle\!\!\right\rangle. \tag{3.20}$$

Therefore the highest weight condition for the singular vector $W_n^a|\chi\rangle = 0$ for $n > 0$ is equivalent to the following loop equation:

Theorem.

$$\left\langle\!\!\left\langle\, \widetilde{\mathcal{W}}_n^a\,\right\rangle\!\!\right\rangle = 0, \qquad n > 0. \tag{3.21}$$

Note that, although the variables $z_j^{(a)}$, z and $x_j^{(a)}$ are all formal parameters, one can treat them as complex parameters with

$$\infty > |x_j^{(a)}|^{-1} > |z| > |z_1^1| > \cdots > |z_{r_1}^1| > \cdots > |z_1^{N-1}| > \cdots > |z_{r_{N-1}}^{N-1}| > 0. \tag{3.22}$$

Here $|z_i^a| > |z_{i+1}^a|$ and $|z_i^a| > |z_j^{a+1}|$.

The quantum spectral curve should be

$$\left\langle\!\!\left\langle\, \left(p^{D_z}-\widetilde{\Lambda}_1(z)\right)\left(p^{D_z}-\widetilde{\Lambda}_2(zp^{-1})\right)\cdots\left(p^{D_z}-\widetilde{\Lambda}_N(zp^{1-N})\right)\,\right\rangle\!\!\right\rangle = 0 \tag{3.23}$$

which regularity in z is guaranteed by the loop equation (3.21).

3.5 Large r_a case

Let $(q,t) =: (e^{R\epsilon_2}, e^{-R\epsilon_1}) =: (e^{g_s R}, e^{g_s \beta R})$ with the radius R of the 5th dimensional circle S^1. Let us rescale the variables as $\tilde{p}_n^{(a)} := g_s p_n^{(a)}$, $\tilde{r}_a := g_s r_a$ and $\tilde{s}_a := g_s s_a$. Then

$$
\begin{aligned}
\widetilde{\Lambda}_i(z) = &\exp\left\{\sum_{n>0} \frac{t^{\frac{n}{2}} - t^{-\frac{n}{2}}}{n g_s} z^n \sum_{b=1}^{N-1} A^{i,b}(p^{-n}) \tilde{p}_n^{(b)}\right\} \\
&\times \exp\left\{\sum_{n>0} \frac{t^{\frac{n}{2}} - t^{-\frac{n}{2}}}{n g_s} z^{-n} \sum_{b=1}^{N-1} B^{i,b}(p^n) \int dw w^n \tilde{r}_b \rho_b(w)\right\} q^{\sqrt{\beta} h^i_{r,s}} p^{\frac{N+1}{2}-i}
\end{aligned} \tag{3.24}
$$

with $\rho_a(w) := \frac{1}{r_a}\sum_{k=1}^{r_a} \delta(w - z_k^a)$. Note that by (3.22),

$$
\begin{aligned}
\sum_{i<k}\left(\frac{z_k^a}{z_i^a}\right)^n - \sum_{j>k}\left(\frac{z_j^a}{z_k^a}\right)^n &= r_a\left[\int_{|w|>|z_k^a|} dw \left(\frac{z_k^a}{w}\right)^n - \int_{|w|<|z_k^a|} dw \left(\frac{w}{z_k^a}\right)^n\right]\rho_a(w), \\
\sum_{n>0}\left\{\sum_{i<k}\left(\frac{z_k^a}{z_i^a}\right)^n - \sum_{j>k}\left(\frac{z_j^a}{z_k^a}\right)^n\right\} &= r_a\left[\int_{|w|>|z_k^a|} dw + \int_{|w|<|z_k^a|} dw\right] \frac{z_k^a \rho_a(w)}{w - z_k^a} + r_a - k \\
&= r_a \fint dw \frac{z_k^a \rho_a(w)}{w - z_k^a} + r_a - k.
\end{aligned} \tag{3.25}
$$

Under the limit $g_s \to 0$ and $r_a,\ s_a,\ k \to \infty$ with fixed $\tilde{r}_a := g_s r_a$, $\tilde{s}_a := g_s s_a$ and $\tilde{k} = g_s k$, the saddle point condition becomes

$$
\begin{aligned}
0 = &\beta \sum_{n>0}\left\{z^n \tilde{p}_n^{(a)} - 2\tilde{r}_a\left[\int_{|w|>|z|} dw \left(\frac{z}{w}\right)^n - \int_{|w|<|z|} dw \left(\frac{w}{z}\right)^n\right]\rho_a(w)\right. \\
&\left. + \tilde{r}_{a-1}\int dw \rho_{a-1}(w)\left(\frac{z}{w}\right)^n - \tilde{r}_{a+1}\int dw \rho_{a+1}(w)\left(\frac{w}{z}\right)^n\right\} + \beta(\tilde{r}_a - 2\tilde{k}) - \tilde{s}_a \\
= &\beta\left[\tilde{r}_{a-1}\int dw \rho_{a-1}(w) - 2\tilde{r}_a \fint dw \rho_a(w) + \tilde{r}_{a+1}\int dw \rho_{a+1}(w)\right]\frac{z}{w-z} \\
&+ \beta \sum_{n>0} z^n \tilde{p}_n^{(a)} + \beta(\tilde{r}_{a+1} - \tilde{r}_a) - \tilde{s}_a
\end{aligned} \tag{3.26}
$$

with $z := \lim_{k\to\infty} z_k$ and $\rho_a(w) := \lim_{r_a \to \infty} \frac{1}{r_a}\sum_{j=1}^{r_a} \delta(w - z_j^a)$. Under this limit, the sift operator p^{D_z} tends to a commutative variable, say $\tilde{z}$, and the spectral curve reduces to

$$
\prod_{i=1}^{N}\left(\tilde{z} - \widetilde{\Lambda}_i(z)\right) = 0, \tag{3.27}
$$

$$
\widetilde{\Lambda}_i(z) = \exp\left\{\beta R\left\{\sum_{n>0} z^n\left[\sum_{a=i}^{N-1} - \sum_{a=1}^{N-1}\frac{a}{N}\right]\tilde{p}_n^{(a)} + \int \frac{dw w}{z-w}\left(\tilde{r}_i \rho_i(w) - \tilde{r}_{i-1}\rho_{i-1}(w)\right)\right\}\right\}
$$

with the solution $\rho_a(w)$ of (3.26). Note that the parameter β appears only in the combinations $\beta \tilde{r}_a$ and $\beta \tilde{p}_n^{(a)}$.

3.6 q-deformed Liouville correlation function

In the relation with the Macdonald polynomial in [21], the parameter p_n is mapped to the power sum $p_n = \sum_{i=1}^{r} x_i^n$. On the other hand, the principal specialization $x_i = t^{i-1}$, i.e., $p_n = (1-t^{rn})/(1-t^n)$, has a natural generalization $p_n = (1-u^n)/(1-t^n)$ with $u \in \mathbb{C}$. By these mapping and specialization, we can identify our partition function with a q-deformed Liouville correlation function.

With parameters $x_j^{(a)}$, $y^{(a)}$ and $u^{(a)}$ for $a = 1, 2, \cdots, N-1$ and $j = 1, 2, \cdots, M_a$, let us consider the case that

$$p_n^{(a)} = \sum_{j=1}^{M_a} (x_j^{(a)})^n + \frac{(u^{(a)})^{\frac{n}{2}} - (u^{(a)})^{-\frac{n}{2}}}{t^{\frac{n}{2}} - t^{-\frac{n}{2}}} (y^{(a)})^n. \tag{3.28}$$

Then

$$V_N = \prod_{a=1}^{N-1} \exp\left\{ \sum_{n>0} \Lambda_n^a \left\{ \sum_{j=1}^{M_a} \frac{(x_j^{(a)})^n}{q^{\frac{n}{2}} - q^{-\frac{n}{2}}} + \frac{(u^{(a)})^{\frac{n}{2}} - (u^{(a)})^{-\frac{n}{2}}}{(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(t^{\frac{n}{2}} - t^{-\frac{n}{2}})} (y^{(a)})^n \right\} \right\} \tag{3.29}$$

is the positive mode part $\left(V_{u^{(a)}}^a(1/y^{(a)})\right)_+$ and $\left(V_+^a(1/x_j^{(a)})\right)_+$ of the primary fields $V_{u^{(a)}}^a(1/y^{(a)})$ in (2.21) and the $(2,1)$ operator $V_+^a(1/x_j^{(a)})$ in (2.26), respectively. Thus the partition function

$$\begin{aligned} Z_N = \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r_a} \frac{dz_j^a}{2\pi i} \cdot \langle \alpha_{r,s}^+ | \prod_{a=1}^{N-1} \left(V_{u^{(a)}}^a(1/y^{(a)})\right)_+ \prod_{j=1}^{M_a} \left(V_+^a(1/x_j^{(a)})\right)_+ \\ \times S_+^1(z_1^1) \cdots S_+^1(z_{r_1}^1) \cdots S_+^{N-1}(z_1^{N-1}) \cdots S_+^{N-1}(z_{r_{N-1}}^{N-1}) | \widetilde{\alpha}_{r,s}^+ \rangle \end{aligned} \tag{3.30}$$

is nothing but the integral part of the q-deformed Liouville correlation function of them,

$$\begin{aligned} &\oint \prod_{a=1}^{N-1} \prod_{j=1}^{r_a} \frac{dz_j^a}{2\pi i} \cdot \langle \alpha_{r,s}^+ | \prod_{a=1}^{N-1} V_{u^{(a)}}^a(1/y^{(a)}) \prod_{a=1}^{N-1} \prod_{j=1}^{M_a} V_+^a(1/x_j^{(a)}) \\ &\qquad \times S_+^1(z_1^1) \cdots S_+^1(z_{r_1}^1) \cdots S_+^{N-1}(z_1^{N-1}) \cdots S_+^{N-1}(z_{r_{N-1}}^{N-1}) | \widetilde{\alpha}_{r,s}^+ \rangle \\ &= f(x,y) Z_N. \end{aligned} \tag{3.31}$$

To recover the hole correlation function, one just need to multiply the OPE factor $f(x,y)$ coming from the negative mode part. Note that if $u^{(a)} = 0$ and $M_a < \infty$ then $\{p_n^{(a)}\}_{n \in \mathbb{N}}$ is linearly dependent and thus q-$\mathcal{W}_N$ constraint (3.16) should be modified but the loop equation (3.21) and the spectral curve (3.23) are unchanged.

4 $N = 2$ case

Here we give an example when $N = 2$, i.e., the q-deformed Virasoro case. The fundamental bosons are

$$[h^1_n, h^1_m] = \frac{1}{n}\frac{(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(t^{\frac{n}{2}} - t^{-\frac{n}{2}})}{p^{\frac{n}{2}} + p^{-\frac{n}{2}}}\delta_{n+m,0}, \qquad [h^1_n, Q^1_h] = \frac{1}{2}\delta_{n,0}, \tag{4.1}$$

$h^2_n := -p^{-n}h^1_n$ and $Q^2_h := -Q^1_h$. The root type and the weight type bosons are $\alpha^1_n := (1+p^{-n})h^1_n$, $Q^1_\alpha := 2Q^1_h$, $\Lambda^1_n := h^1_n p^{-\frac{n}{2}}$ and $Q^1_\Lambda := Q^1_h$. Note that $A^{1,1}(p) = 1/(1+p^{-1})$, $B^{1,1}(p) = p^{\frac{1}{2}}$ and $C^{1,1}(p) = [\,2\,]_p$.

The q-Virasoro generator, the screening currents and the vertex operators are now

$$\begin{aligned}
W^1(z) &= :\exp\left\{\sum_{n\neq 0} h^1_n z^{-n}\right\}: q^{\sqrt{\beta}h^1_0} p^{\frac{1}{2}} + :\exp\left\{-\sum_{n\neq 0} h^1_n p^{-n} z^{-n}\right\}: q^{-\sqrt{\beta}h^1_0} p^{-\frac{1}{2}},\\
S^1_\pm(z) &= :\exp\left\{\mp\sum_{n\neq 0} \frac{1+p^{-n}}{\xi_\pm^{\frac{n}{2}} - \xi_\pm^{-\frac{n}{2}}} h^1_n z^{-n}\right\}: e^{\pm 2\sqrt{\beta}^{\pm 1} Q^1_h} z^{\pm 2\sqrt{\beta}^{\pm 1} h^1_0}, \qquad \xi_+ = q, \quad \xi_- = t,\\
V^1_\pm(z) &= :\exp\left\{\pm\sum_{n\neq 0} \frac{h^1_n}{\xi_\pm^{\frac{n}{2}} - \xi_\pm^{-\frac{n}{2}}} p^{-\frac{n}{2}} z^{-n}\right\}: e^{\mp\sqrt{\beta}^{\pm 1} Q^1_h} z^{\mp\sqrt{\beta}^{\pm 1} h^1_0},\\
V^1_u(z) &= :\exp\left\{\sum_{n\neq 0} \frac{(u^{\frac{n}{2}} - u^{-\frac{n}{2}})h^1_n}{(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(t^{\frac{n}{2}} - t^{-\frac{n}{2}})} p^{-\frac{n}{2}} z^{-n}\right\}: e^{-\gamma\sqrt{\beta}Q^1_h} z^{-\gamma\sqrt{\beta}h^1_0},\\
V_2 &= \exp\left\{\sum_{n>0} \frac{h^1_n}{q^{\frac{n}{2}} - q^{-\frac{n}{2}}} p^{-\frac{n}{2}} p_n\right\}.
\end{aligned} \tag{4.2}$$

For non-negative integers s and $r \geq 0$, the singular vectors $|\chi_{rs}\rangle \in \mathcal{F}_{\alpha_{rs}}$ are

$$\begin{aligned}
|\chi^+_{r,s}\rangle &= \oint \prod_{j=1}^r \frac{dz_j}{2\pi i} \cdot S^1_+(z_1)\cdots S^1_+(z_r)|\widetilde{\alpha}^+_{r,s}\rangle\\
&= \oint \prod_{j=1}^r \frac{dz_j}{2\pi i z_j} z_j^{-s}(S^1_+(z_j))_- \cdot \Delta^{qW}(z)|\alpha^+_{r,s}\rangle
\end{aligned} \tag{4.3}$$

with $\alpha^{+,1}_{r,s} := \sqrt{\beta}(1+r) - \frac{1}{\sqrt{\beta}}(1+s)$. Here $\Delta^{qW}(z)$ is same as (2.31). The partition function Z_2 is now

$$\begin{aligned}
Z_2(p) &= \oint \prod_{j=1}^r \frac{dz_j}{2\pi i z_j} z_j^{-s} \exp\left\{\sum_{n>0} \frac{1}{n}\frac{t^{\frac{n}{2}} - t^{-\frac{n}{2}}}{q^{\frac{n}{2}} - q^{-\frac{n}{2}}} z_j^n p_n\right\} \cdot \Delta^{qW}(z)\\
&= p^{\frac{rs}{2}} \frac{r!\langle\, s^r \,\rangle''_r}{\langle\, s^r \,\rangle} P_{(s^r)}(x[\,p\,]).
\end{aligned} \tag{4.4}$$

If we divide the power sum p_n as $p_n =: p_n^{(1)} + p_n^{(2)}$ then by (A.13) and (A.16) we have

$$
\begin{aligned}
Z_2(p^{(1)}+p^{(2)}) &= p^{\frac{rs}{2}} \sum_{\lambda\subset(s^r)} \frac{r!\langle\, s^r \,\rangle''_r}{\langle\, \lambda \,\rangle\langle\, \widehat{\lambda} \,\rangle} f^{(s^r)}_{\lambda,\widehat{\lambda}} P_\lambda(x[\, p^{(1)}\,]) P_{\widehat{\lambda}}(x[\, p^{(2)}\,]) \\
&= p^{\frac{rs}{2}} \frac{r!\langle\, s^r \,\rangle''_r}{\langle\, s^r \,\rangle'_r} \sum_{\lambda\subset(s^r)} \frac{\langle\, \lambda \,\rangle'_r}{\langle\, \lambda \,\rangle\langle\, \widehat{\lambda} \,\rangle} P_\lambda(x[\, p^{(1)}\,]) P_{\widehat{\lambda}}(x[\, p^{(2)}\,]). \qquad (4.5)
\end{aligned}
$$

Here $\lambda \subset (s^r)$ means $\lambda_1 \leq s$ and $\ell(\lambda) \leq r$.

Let us consider the case that $p_n^{(2)} := (1-u^n)y^n/(1-t^n)$ and denote its function by $f(x[\, \frac{1-u}{1-t} y\,])$. Then we have

$$
\frac{P_{\widehat{\lambda}}\left(x[\, \frac{1-u}{1-t}y \,]\right)}{\langle\, \widehat{\lambda} \,\rangle y^{|\widehat{\lambda}|}} = \frac{P_{(s^r)}\left(x[\, \frac{1-u}{1-t}y \,]\right)}{\langle\, s^r \,\rangle y^{rs}} \prod_{(i,j)\in\lambda} \frac{(1-q^j t^{r-i})(1-q^{s-j+1}t^{i-1})}{(t^{r-i}-uq^{s-j})(1-q^{\lambda_i-j+1}t^{\lambda'_j-i})}, \qquad (4.6)
$$

which is proved in the appendix B. Next, (A.6) shows that

$$
\frac{\langle\, s^r \,\rangle}{\langle\, s^r \,\rangle'_r} \frac{\langle\, \lambda \,\rangle'_r}{\langle\, \lambda \,\rangle} = \prod_{(i,j)\in(s^r)} \frac{1-q^j t^{r-i}}{1-q^{j-1}t^{r-i+1}} \prod_{(i,j)\in\lambda} \frac{1-q^{j-1}t^{r-i+1}}{1-q^j t^{r-i}}. \qquad (4.7)
$$

From the above two equations,

$$
\frac{1}{\langle\, s^r \,\rangle'_r} \frac{\langle\, \lambda \,\rangle'_r}{\langle\, \lambda \,\rangle} \frac{P_{\widehat{\lambda}}\left(x[\, \frac{1-u}{1-t}y \,]\right)}{\langle\, \widehat{\lambda} \,\rangle y^{|\widehat{\lambda}|}} = \frac{P_{(s^r)}\left(x[\, \frac{1-u}{1-t}y \,]\right)}{\langle\, s^r \,\rangle y^{rs}} \prod_{(i,j)\in\lambda} \frac{(1-q^{j-1}t^{r-i+1})(1-q^{s-j+1}t^{i-1})}{(t^{r-i}-uq^{s-j})(1-q^{\lambda_i-j+1}t^{\lambda'_j-i})}, \qquad (4.8)
$$

hence we obtain

$$
\begin{aligned}
\frac{\langle\, s^r \,\rangle}{r!\langle\, s^r \,\rangle''_r} Z_2\left(p^{(1)} + \frac{1-u}{1-t}y\right) &= p^{\frac{rs}{2}} P_{(s^r)}\left(x\left[\frac{1-u}{1-t}y\right]\right) \\
&\times \sum_{\lambda\subset(s^r)} y^{-|\lambda|} P_\lambda(x[\, p^{(1)}\,]) \prod_{(i,j)\in\lambda} \frac{q}{u} \frac{(t^{i-1}-q^{j-1}t^r)(t^{i-1}-q^{j-1-s})}{(t^{i-1}-t^{r-1}q^{j-s}/u)(1-q^{\lambda_i-j+1}t^{\lambda'_j-i})}. \qquad (4.9)
\end{aligned}
$$

The involution $\omega_{q,t}$ is defined in (A.10) as $\omega_{q,t}(p_n) = (-1)^{n-1}p_n(1-q^n)/(1-t^n)$. If we act $\omega_{q,t}$ on the variables p_n in (4.4) and denote it as $\omega_{q,t}Z_2(p^{(0)}) := \omega_{q,t}Z_2(p)|_{p=p^{(0)}}$, then we get another type of formula

$$
\begin{aligned}
\frac{1}{r!\langle\, s^r \,\rangle''_r}\omega_{q,t}Z_2(p) &= \frac{1}{r!\langle\, s^r \,\rangle''_r} \oint \prod_{j=1}^{r} \frac{dz_j}{2\pi i z_j} z_j^{-s} \exp\left\{\sum_{n>0} \frac{(-1)^{n-1}}{n} v^n z_j^n p_n\right\} \cdot \Delta^{qW}(z) \\
&= p^{\frac{rs}{2}} P_{(r^s)}(x[\, p\,]; t, q) \\
&= \frac{\langle\, s^r \,\rangle}{r!\langle\, s^r \,\rangle''_r} Z_2(p)|_{\substack{q\leftrightarrow t \\ r\leftrightarrow s}} \qquad (4.10)
\end{aligned}
$$

which is just (4.4) with the replacement $q \leftrightarrow t$ and $r \leftrightarrow s$.

Therefore when $p_n^{(1)} := \sum_{i=1}^M x_i^n$ we obtain

Proposition. *The partition function $Z_2(p)$ substituting $p_n = \sum_i x_i^n + \frac{1-u^n}{1-t^n} y^n$ and $\frac{1-t^n}{1-q^n} p_n = (-1)^{n-1}(\sum_i x_i^n + \frac{1-u^n}{1-t^n} y^n)$ are*

$$\frac{Z_2\left(\sum_i x_i + \frac{1-u}{1-t} y\right)}{Z_2\left(\frac{1-u}{1-t} y\right)} = {}_2\varphi_1^{(q,t)}\left[\begin{matrix} q^{-s}, t^r \\ q^{1-s}t^{r-1}/u \end{matrix}; \frac{qx}{uy}\right], \tag{4.11}$$

$$\omega_{q,t} \frac{Z_2\left(\sum_i x_i + \frac{1-u}{1-t} y\right)}{Z_2\left(\frac{1-u}{1-t} y\right)} = {}_2\varphi_1^{(t,q)}\left[\begin{matrix} t^{-r}, q^s \\ t^{1-r}q^{s-1}/u \end{matrix}; \frac{tx}{uy}\right]. \tag{4.12}$$

Here ${}_2\varphi_1^{(q,t)}\left[\begin{smallmatrix} a,b \\ c \end{smallmatrix}; x\right]$ is the multivariate q-hypergeometric function [33]

$${}_2\varphi_1^{(q,t)}\left[\begin{matrix} a, b \\ c \end{matrix}; x\right] := \sum_{\substack{\lambda \\ \ell(\lambda) \le M}} P_\lambda(x) \prod_{(i,j)\in\lambda} \frac{(t^{i-1} - aq^{j-1})(t^{i-1} - bq^{j-1})}{(t^{i-1} - cq^{j-1})(1 - q^{\lambda_i - j + 1} t^{\lambda'_j - i})}. \tag{4.13}$$

Since $P_\lambda(x; q, t) = P_\lambda(x; q^{-1}, t^{-1})$, ${}_2\varphi_1^{(q,t)}\left[\begin{smallmatrix} a,b \\ c \end{smallmatrix}; x\right]$ satisfies

$${}_2\varphi_1^{(q,t)}\left[\begin{matrix} a, b \\ c \end{matrix}; x\right] = {}_2\varphi_1^{(q^{-1},t^{-1})}\left[\begin{matrix} a^{-1}, b^{-1} \\ c^{-1} \end{matrix}; \frac{ab}{qc} x\right]. \tag{4.14}$$

When $M = \infty$,

$$\omega_{q,t} {}_2\varphi_1^{(q,t)}\left[\begin{matrix} a, b \\ c \end{matrix}; x\right] = {}_2\varphi_1^{(t,q)}\left[\begin{matrix} a, b \\ c \end{matrix}; \frac{ab}{c} x\right], \qquad M = \infty. \tag{4.15}$$

When $M = 1$, ${}_2\varphi_1^{(q,t)}\left[\begin{smallmatrix} a,b \\ c \end{smallmatrix}; x\right]$ reduces to the usual q-hypergeometric function

$${}_2\varphi_1^{(q,t)}\left[\begin{matrix} a, b \\ c \end{matrix}; x\right] := {}_2\varphi_1\left[\begin{matrix} a, b \\ c \end{matrix}; q, x\right] := \sum_{n \ge 0} x^n \prod_{\ell=0}^{n-1} \frac{(1 - aq^\ell)(1 - bq^\ell)}{(1 - cq^\ell)(1 - q^{\ell+1})}, \qquad M = 1. \tag{4.16}$$

In the next section we will show a relation between our $Z_2\left(x + \frac{1-u}{1-t} y\right)$ and the 5-dimensional $SU(2)$ Nekrasov partition function.

5 Five-dimensional Nekrasov partition function

Let $Q = (Q_1, \cdots, Q_N)$ and $Q^\pm = (Q_1^\pm, \cdots, Q_N^\pm)$ be sets of complex parameters. The instanton part of the five-dimensional $SU(N)$ Nekrasov partition function with $N_f = 2N$

fundamental matters[3] is written by a sum over N Young diagrams λ_i $(i = 1, 2, \cdots, N)$ as follows [26][17]:[4]

$$Z^{\text{inst}}(Q) := \sum_{\{\lambda_i\}} \prod_{i,j} \frac{\sqrt{\prod_{\epsilon=\pm} N_{\lambda_i \bullet}(vQ_i/Q_j^{\epsilon}) N_{\bullet \lambda_i}(vQ_j^{\epsilon}/Q_i)}}{N_{\lambda_i \lambda_j}(Q_i/Q_j)} \cdot \prod_i \left(\frac{\Lambda^2}{v}\right)^{N|\lambda_i|} \tag{5.1}$$

with $v := (q/t)^{\frac{1}{2}}$ and

$$\begin{aligned} N_{\lambda\mu}(Q) := N_{\lambda\mu}(Q;q,t) &:= \prod_{(i,j)\in\lambda} \left(1 - Q\, q^{\lambda_i - j} t^{\mu'_j - i + 1}\right) \prod_{(i,j)\in\mu} \left(1 - Q\, q^{-\mu_i + j - 1} t^{-\lambda'_j + i}\right) \\ &= \prod_{(i,j)\in\mu} \left(1 - Q\, q^{\lambda_i - j} t^{\mu'_j - i + 1}\right) \prod_{(i,j)\in\lambda} \left(1 - Q\, q^{-\mu_i + j - 1} t^{-\lambda'_j + i}\right) \end{aligned} \tag{5.2}$$

Here $\lambda = (\lambda_1, \lambda_2, \cdots)$ is a Young diagram such that $\lambda_i \geq \lambda_{i+1}$. λ' is its conjugate Young diagram and $|\lambda| = \sum_i \lambda_i$. $Z^{\text{inst}}(Q; Q^+; Q^-)$ is symmetric in masses ${Q^{\pm}}_j$'s. Note that $N_{\lambda\mu}(Q; q, t)$ satisfies

$$N_{\lambda\mu}(vQ; q, t) = N_{\mu\lambda}(Q/v; q^{-1}, t^{-1}) = N_{\mu'\lambda'}(Q/v; t, q) \tag{5.3}$$

and

$$N_{\lambda\bullet}(vQ) N_{\bullet\lambda}(vQ') = N_{\bullet\lambda}(v/Q) N_{\lambda\bullet}(v/Q') (QQ')^{|\lambda|}. \tag{5.4}$$

Using (5.4), (5.1) is rewritten to the following two ways (double-sign corresponds):

$$Z^{\text{inst}}(Q) = \sum_{\{\lambda_i\}} \prod_{i,j} \frac{N_{\lambda_i \bullet}(vQ_i/Q_j^{\pm}) N_{\bullet \lambda_i}(vQ_j^{\mp}/Q_i)}{N_{\lambda_i \lambda_j}(Q_i/Q_j)} \cdot \prod_i \left(\frac{\Lambda_\alpha^{\pm}}{v^N}\right)^{|\lambda_i|} \tag{5.5}$$

with

$$\Lambda_\alpha^{\pm} := \Lambda^{2N} \prod_{j=1}^{N} \left(\frac{Q_j^{\pm}}{Q_j^{\mp}}\right)^{\frac{1}{2}}. \tag{5.6}$$

[3]The parameters (q, t) are related with those (ϵ_1, ϵ_2) of the Ω background through $(q, t) = (e^{R\epsilon_2}, e^{-R\epsilon_1})$ where R is the radius of the 5th dimensional circle. The parameter Q is related with the vacuum expectation value a of the scalar fields in the vector multiplets and the mass m of the fundamental matter as $Q_i = q^{a_i}$, $Q_i^+ = q^{-m_i}$ and $Q_i^- = q^{-m_{N+i}}$.

[4]In [17], there are typos in (9.4) and (9.5). For $\alpha < \beta$, $Q_{\alpha,\beta}$ and $Q'_{\alpha,\beta}$ should be replaced with $v^{1+\frac{(-1)^\alpha + (-1)^\beta}{2}} Q_{\alpha,\beta}$ and $v^{-1-\frac{(-1)^\alpha + (-1)^\beta}{2}} Q'_{\alpha,\beta}$, respectively. Furthermore we change the definition of Λ_α in [17] as follows

$$\Lambda_\alpha := \Lambda^{2N} \prod_{\beta=1}^{\alpha-1} \frac{Q_\beta}{\sqrt{Q_\beta^+ Q_\beta^-}} \prod_{\beta=\alpha}^{N} \frac{\sqrt{Q_\beta^+ Q_\beta^-}}{Q_\beta}.$$

Note that for λ or $\mu = (0)$,

$$N_{\lambda\bullet}(Q) = \prod_{(i,j)\in\lambda} (1 - Q\, q^{j-1} t^{1-i}), \quad N_{\bullet\lambda}(Q) = \prod_{(i,j)\in\lambda} (1 - Q\, q^{-j} t^{i}) \tag{5.7}$$

and $N_{\bullet\bullet}(Q) = 1$. Here $\bullet$ denotes (0). Hence for a special value of Q, $N_{\lambda\bullet}(1) = \delta_{\lambda,\bullet}$ and

$$N_{\lambda\bullet}(t) = \sum_{n\geq 0} \delta_{\lambda,n} \prod_{\ell=0}^{n-1} (1 - q^{\ell} t), \qquad N_{\lambda\bullet}(1/q) = \sum_{n\geq 0} \delta_{\lambda,1^n} \prod_{\ell=0}^{n-1} (1 - t^{-\ell}/q). \tag{5.8}$$

Therefore one can adjust the parameter Q so that a factor of numerator of (5.1), $N_{\lambda\bullet}(Q)$, vanishes except for $\lambda = (0)$, (n) or (1^n). Namely for some j, if $vQ_i/Q_j^{\pm} = 1$, t or q^{-1} then the right hand side of (5.1) is summed over only $\lambda_i = (0)$, (n) or (1^n) with $n \in \mathbb{Z}_{\geq 0}$, respectively. Note also that for λ, $\mu = (0)$, (n) or (1^n),

$$\begin{aligned}
N_{nn}(Q) = \prod_{\ell=0}^{n-1} (1 - Q\, q^{\ell} t)(1 - Q\, q^{-\ell-1}), &\quad N_{1^n 1^n}(Q) = \prod_{\ell=0}^{n-1} (1 - Q\, t^{\ell+1})(1 - Q\, t^{-\ell}/q), \\
N_{n\bullet}(Q) = \prod_{\ell=0}^{n-1} (1 - Q\, q^{\ell}), &\quad N_{\bullet n}(Q) = \prod_{\ell=0}^{n-1} (1 - Q\, q^{-\ell-1} t), \\
N_{1^n\bullet}(Q) = \prod_{\ell=0}^{n-1} (1 - Q\, t^{-\ell}), &\quad N_{\bullet 1^n}(Q) = \prod_{\ell=0}^{n-1} (1 - Q\, t^{\ell+1}/q)
\end{aligned} \tag{5.9}$$

and

$$\frac{N_{n\bullet}(t)}{N_{nn}(1)} = \prod_{\ell=0}^{n-1} \frac{1}{1 - q^{-\ell-1}}, \qquad \frac{N_{1^n\bullet}(1/q)}{N_{1^n 1^n}(1)} = \prod_{\ell=0}^{n-1} \frac{1}{1 - t^{\ell+1}}. \tag{5.10}$$

Hence one can adjust N out of $N_f = 2N$ parameters $Q^{\pm}{}_i$'s so that (5.5) reduces to all $\lambda_i = (0)$ but a $\lambda_j = (n)$ or (1^n) with $n \in \mathbb{Z}_{\geq 0}$ same as [27]. For example, if $(Q_1, \cdots, Q_{N-1}, Q_N) = v^{-1} \times (Q_1^{\pm}, \cdots, Q_{N-1}^{\pm}, tQ_N^{\pm})$ then the right hand side of (5.5) is summed over only $(\lambda_1, \cdots, \lambda_{N-1}, \lambda_N) = ((0), \cdots, (0), (n))$ with $n \in \mathbb{Z}_{\geq 0}$ and thus

$$\begin{aligned}
&Z^{\text{inst}}(Q_1^{\pm}/v, \cdots, Q_{N-1}^{\pm}/v, tQ_N^{\pm}/v) \\
&= \sum_{n\geq 0} \left(\frac{\Lambda_N^{\pm}}{v^N}\right)^n \frac{N_{n\bullet}(vQ_N/Q_N^{\pm}) N_{\bullet n}(vQ_N^{\mp}/Q_N)}{N_{nn}(Q_N/Q_N)} \prod_{j=1}^{N-1} \frac{N_{n\bullet}(vQ_N/Q_j^{\pm})}{N_{n\bullet}(Q_N/Q_j)} \frac{N_{\bullet n}(vQ_j^{\mp}/Q_N)}{N_{\bullet n}(Q_j/Q_N)} \\
&= \sum_{n\geq 0} \left(\frac{\Lambda_N^{\pm}}{v^N}\right)^n \prod_{\ell=0}^{n-1} \frac{1 - q^{-\ell} Q_N/vQ_N^{\mp}}{1 - q^{-\ell-1}} \prod_{j=1}^{N-1} \frac{1 - q^{-\ell} Q_N/vQ_j^{\mp}}{1 - tq^{-\ell-1} Q_N/Q_j}.
\end{aligned} \tag{5.11}$$

On the other hand, if $(Q_1, \cdots, Q_{N-1}, Q_N) = v^{-1} \times (Q_1^{\pm}, \cdots, Q_{N-1}^{\pm}, Q_N^{\pm}/q)$ then only $(\lambda_1, \cdots, \lambda_{N-1}, \lambda_N) = ((0), \cdots, (0), (1^n))$ contributes. Therefore we obtain

Proposition.

$$
\begin{aligned}
Z^{\text{inst}}(Q_1^{\pm}/v,\cdots,Q_{N-1}^{\pm}/v,tQ_N^{\pm}/v) &= {}_N\varphi_{N-1}\left[\begin{matrix}\frac{Q_1^{\mp}}{vQ_N},\cdots,\frac{Q_N^{\mp}}{vQ_N}\\ \frac{tQ_1}{qQ_N},\cdots,\frac{tQ_{N-1}}{qQ_N}\end{matrix};q^{-1},\frac{\Lambda_N^{\pm}}{v^N}\right]\\
&= {}_N\varphi_{N-1}\left[\begin{matrix}v\frac{Q_N}{Q_1^{\mp}},\cdots,v\frac{Q_N}{Q_N^{\mp}}\\ \frac{qQ_N}{tQ_1},\cdots,\frac{qQ_N}{tQ_{N-1}}\end{matrix};q,v^N\Lambda_N^{\mp}\right],\\
Z^{\text{inst}}(Q_1^{\pm}/v,\cdots,Q_{N-1}^{\pm}/v,Q_N^{\pm}/qv) &= {}_N\varphi_{N-1}\left[\begin{matrix}\frac{Q_1^{\mp}}{vQ_N},\cdots,\frac{Q_N^{\mp}}{vQ_N}\\ \frac{tQ_1}{qQ_N},\cdots,\frac{tQ_{N-1}}{qQ_N}\end{matrix};t,\frac{\Lambda_N^{\pm}}{v^N}\right]
\end{aligned}
\tag{5.12}
$$

with

$$
{}_r\varphi_s\left[\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_s\end{matrix};q,x\right]:=\sum_{n\geq 0}x^n\prod_{\ell=0}^{n-1}\frac{(-q^{\ell})^{s+1-r}\prod_{i=1}^r(1-q^{\ell}a_i)}{(1-q^{\ell+1})\prod_{i=1}^s(1-q^{\ell}b_i)}. \tag{5.13}
$$

Note that

$$
{}_r\varphi_{r-1}\left[\begin{matrix}a_1,\cdots,a_r\\ b_1,\cdots,b_{r-1}\end{matrix};q,x\right]={}_r\varphi_{r-1}\left[\begin{matrix}a_1^{-1},\cdots,a_r^{-1}\\ b_1^{-1},\cdots,b_{r-1}^{-1}\end{matrix};q^{-1},\widetilde{x}\right],\qquad \widetilde{x}:=\frac{x\prod_{i=1}^r a_i}{q\prod_{i=1}^{r-1}b_i}. \tag{5.14}
$$

When $N=2$, Z^{inst} coincides with the $M=1$ case of the partition function Z_2 of the q-deformed β-ensemble (4.11) similar to [6]

$$
Z^{\text{inst}}(Q_1^{\pm}/v,tQ_2^{\pm}/v)={}_2\varphi_1\left[\begin{matrix}v\frac{Q_2}{Q_1^{\mp}},v\frac{Q_2}{Q_2^{\mp}}\\ \frac{qQ_2}{tQ_1}\end{matrix};q,v^2\Lambda_2^{\mp}\right]=\frac{Z_2\left(x+\frac{1-u}{1-t}y\right)}{Z_2\left(\frac{1-u}{1-t}y\right)}, \tag{5.15}
$$

$$
Z^{\text{inst}}(Q_1^{\pm}/v,Q_2^{\pm}/qv)={}_2\varphi_1\left[\begin{matrix}\frac{Q_1^{\mp}}{vQ_2},\frac{Q_2^{\mp}}{vQ_2}\\ \frac{tQ_1}{qQ_2}\end{matrix};t,\frac{\Lambda_2^{\pm}}{v^2}\right]=\omega_{q,t}\frac{Z_2\left(\sum_i x_i+\frac{1-u}{1-t}y\right)}{Z_2\left(\frac{1-u}{1-t}y\right)} \tag{5.16}
$$

with

$$
q^s=\frac{Q_1^{\mp}}{vQ_2},\quad t^{-r}=\frac{Q_2^{\mp}}{vQ_2},\quad u=\frac{qQ_1Q_2}{tQ_1^{\mp}Q_2^{\mp}},\quad \frac{qx}{y}=\frac{Q_1^{\mp}Q_2^{\mp}}{Q_1Q_2}\Lambda_2^{\mp} \tag{5.17}
$$

for (5.15) and

$$
q^{-s}=v\frac{Q_2}{Q_1^{\mp}},\quad t^{r}=v\frac{Q_2}{Q_2^{\mp}},\quad u=\frac{tQ_1^{\mp}Q_2^{\mp}}{qQ_1Q_2},\quad \frac{tx}{y}=\frac{Q_1Q_2}{Q_1^{\mp}Q_2^{\mp}}\Lambda_2^{\pm} \tag{5.18}
$$

for (5.16). In the $SU(N)$ case, the Nekrasov partition function (5.12) may coincide with our partition function Z_N by using the formulas (D.5) and (A.20).

Acknowledgments

We would like thank H. Fuji, J. Kaneko, H. Kanno, M. Manabe, H. Ochiai, S. Odake, J. Shiraishi, S. Warnaar and S. Yanagida for discussions. The work of Y.Y. is supported in part by Grant-in-Aid for Scientific Research [#21340036] from the Japan Ministry of Education, Culture, Sports, Science and Technology. The work of H.A. is supported in part by Daiko Foundation.

Appendix A: Macdonald polynomial

Here we recapitulate basic properties of the Macdonald polynomial [24]. Let $\lambda := (\lambda_1, \lambda_2, \cdots, \lambda_r)$ with $\lambda_i \geq \lambda_{i+1} \geq 0$ be a Young diagram. λ' is its conjugate. For any λ with $\lambda_1 \leq s$, $\widehat{\lambda}$ is complements of λ with respect to (s^r), i.e., $\widehat{\lambda}_i = s - \lambda_{r-i+1}$. $|\lambda| := \sum_i \lambda_i$. Let $x := (x_1, \cdots, x_r)$ and $p := (p_1, p_2, \cdots)$ with the power sum $p_n := p_n(x) := \sum_{i=1}^r x_i^n$. For any symmetric function f in x with $r = \infty$, $f(x[p])$ stands for the function f expressed in the power sum p.

The Macdonald polynomials $P_\lambda(x) := P_\lambda(x; q, t)$ are degree $|\lambda|$ homogeneous symmetric polynomials in x defined as eigenfunctions of the Macdonald operator H as follows:

$$
\begin{aligned}
&H P_\lambda(x) = \varepsilon_\lambda P_\lambda(x), \\
&H := \sum_{i=1}^r \prod_{j(\neq i)} \frac{t x_i - x_j}{x_i - x_j} \cdot q^{D_{x_i}}, \qquad \varepsilon_\lambda := \sum_{i=1}^r q^{\lambda_i} t^{r-i}
\end{aligned}
\tag{A.1}
$$

with a normalization condition $P_\lambda(x) = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_r^{\lambda_r} + \cdots$. Where q^{D_x} with $D_x := x\frac{\partial}{\partial x}$ is the q-shift operator such that $q^{D_x} f(x) = f(qx)$. Note that $P_\bullet(x) := P_{(0)}(x) = 1$.

Two kinds of inner products are known in which the Macdonald polynomials are orthogonal each other. For any symmetric functions f and g in x, let us define inner product $\langle *, * \rangle$ and another one $\langle *, * \rangle'_r$ as follows:

$$
\langle f, g \rangle := \oint \prod_{n>0} \frac{dp_n}{2\pi i p_n} \cdot f(x[p^*])\, g(x[p]), \qquad p_n^* := n \frac{1-q^n}{1-t^n} \frac{\partial}{\partial p_n}, \tag{A.2}
$$

$$
\langle f, g \rangle'_r := \frac{1}{r!} \oint \prod_{j=1}^r \frac{dx_j}{2\pi i x_j} \cdot \Delta^{Mac}(x)\, f(\overline{x})\, g(x), \qquad \overline{x_j} := \frac{1}{x_j} \tag{A.3}
$$

with

$$
\Delta^{Mac}(x) := \prod_{i \neq j}^r \exp\left\{ -\sum_{n>0} \frac{1}{n} \frac{1-t^n}{1-q^n} \frac{x_j^n}{x_i^n} \right\} = \prod_{i\neq j}^r \prod_{\ell \geq 0} \frac{1 - q^\ell x_j / x_i}{1 - t q^\ell x_j / x_i}, \qquad |q| < 1. \tag{A.4}
$$

Here we must treat the power sums p_n as formally independent variables, *i.e.*, $\frac{\partial}{\partial p_n} p_m = \delta_{n,m}$ for all $n, m > 0$. The inner products of Macdonald polynomials are given by

$$\langle P_\lambda, P_\mu \rangle = \delta_{\lambda,\mu} \langle \lambda \rangle, \qquad \langle \lambda \rangle := \prod_{(i,j)\in\lambda} \frac{1 - q^{\lambda_i - j+1} t^{\lambda'_j - i}}{1 - q^{\lambda_i - j} t^{\lambda'_j - i+1}}, \tag{A.5}$$

$$\langle P_\lambda, P_\mu \rangle'_r = \delta_{\lambda,\mu} \langle \lambda \rangle'_r, \quad \frac{\langle \lambda \rangle'_r}{\langle \lambda \rangle} := \prod_{(i,j)\in\lambda} \frac{1 - q^{j-1} t^{r-i+1}}{1 - q^j t^{r-i}} \prod_{k=1}^{r} \frac{\Gamma_q(k\beta)}{\Gamma_q(\beta)\Gamma_q((k-1)\beta+1)} \tag{A.6}$$

Here $\Gamma_q(x)$ is the q-deformed Γ function $\Gamma_q(x) := (1-q)^{1-x} \prod_{\ell\geq 0} \frac{1-q^{\ell+1}}{1-q^{\ell+x}}$. Since the Macdonald operator is self-adjoint for the another inner product $\langle *, * \rangle'_r$, that is to say $\langle H\, f, g \rangle'_r = \langle f, H\, g \rangle'_r$ (eq. (VI.9.4) in [24]), the Macdonald polynomials are orthogonal for this product $\langle P_\lambda, C\, P_\mu \rangle'_r \propto \delta_{\lambda,\mu}$ with an arbitrary pseudo-constant $C(x)$, i.e., $q^{D_{x_i}} C(x) = C(x)$. Since $\Delta^{qW}(x)/\Delta^{Mac}(x)$ is a pseudo-constant, the other inner product replacing $\Delta^{Mac}(x)$ with $\Delta^{qW}(x)$ as

$$\langle f, g \rangle''_r := \frac{1}{r!} \oint \prod_{j=1}^{r} \frac{dx_j}{2\pi i x_j} \cdot \Delta^{qW}(x)\, f(\overline{x})\, g(x), \qquad \overline{x_j} := \frac{1}{x_j} \tag{A.7}$$

also has orthogonality, i.e., $\langle P_\lambda, P_\mu \rangle''_r = \delta_{\lambda,\mu} \langle \lambda \rangle''_r$. Let us denote by $f(x\left[\frac{1-u}{1-t}\right])$ the function $f(x[p])$ in the specialization $p_n := (1-u^n)/(1-t^n)$ with $u \in \mathbb{C}$, then [24]

$$P_\lambda \left(x\left[\frac{1-u}{1-t} \right] \right) = \prod_{(i,j)\in\lambda} \frac{t^{i-1} - u q^{j-1}}{1 - q^{\lambda_i - j} t^{\lambda'_j - i+1}}. \tag{A.8}$$

The following Cauchy formula is especially important:

$$\sum_\lambda \frac{1}{\langle \lambda \rangle} P_\lambda(x;q,t) P_\lambda(y;q,t) = \Pi(x,y) := \exp\left\{ \sum_{n>0} \frac{1}{n} \frac{1-t^n}{1-q^n} p_n(x) p_n(y) \right\}. \tag{A.9}$$

With the involution $\omega_{q,t}$,

$$\frac{1}{\langle \lambda \rangle} \omega_{q,t} P_\lambda(x;q,t) = P_{\lambda'}(x;t,q), \qquad \omega_{q,t}(p_n) = (-1)^{n-1} \frac{1-q^n}{1-t^n} p_n. \tag{A.10}$$

If we act $\omega_{q,t}$ on x of $\Pi(x,y)$, it becomes

$$\Pi_0(x,y) := \exp\left\{ \sum_{n>0} \frac{(-1)^{n-1}}{n} p_n(x) p_n(y) \right\}. \tag{A.11}$$

Let us denote a symmetric function f in the set of variables $(x_1, x_2, \cdots, y_1, y_2, \cdots)$ by $f\,(x,y)$ or $f\,(\{x,y\})$. Let $f^\nu_{\lambda,\mu}$ be the following fusion coefficient

$$P_\lambda(x) P_\mu(x) =: \sum_\nu f^\nu_{\lambda,\mu} P_\nu(x), \tag{A.12}$$

i.e., $f^{\nu}_{\lambda,\mu} := \langle P_\lambda P_\mu, P_\nu\rangle / \langle P_\nu, P_\nu\rangle$. Then we have

Lemma.
$$\frac{P_\nu(x,y)}{\langle\, \nu\,\rangle} = \sum_{\substack{\lambda,\mu \\ \lambda,\mu\subset\nu}} \frac{P_\lambda(x)}{\langle\, \lambda\,\rangle} f^{\nu}_{\lambda,\mu} \frac{P_\mu(y)}{\langle\, \mu\,\rangle}. \tag{A.13}$$

Proof. By the Cauchy formula (A.9),

$$\begin{aligned}\sum_\nu \frac{P_\nu(x,y)P_\nu(z)}{\langle\, \nu\,\rangle} &= \Pi(\{x,y\},z) = \Pi(x,z)\Pi(y,z)\\ &= \sum_{\lambda,\mu} \frac{P_\lambda(x)P_\lambda(z)}{\langle\, \lambda\,\rangle} \frac{P_\mu(z)P_\mu(y)}{\langle\, \mu\,\rangle}\\ &= \sum_{\lambda,\mu,\nu} \frac{P_\lambda(x)}{\langle\, \lambda\,\rangle} f^{\nu}_{\lambda,\mu} P_\nu(z) \frac{P_\mu(y)}{\langle\, \mu\,\rangle}. \end{aligned}\tag{A.14}$$

□

The fusion coefficient satisfies [28]

$$f^{\nu}_{\lambda,\mu} = f^{\widehat{\lambda}}_{\mu,\widehat{\nu}} \frac{\langle\, \lambda\,\rangle'_r}{\langle\, \nu\,\rangle'_r} \tag{A.15}$$

where $\widehat{\lambda}$ is the complements of λ with respect to (s^r) with $\lambda_1 \le s$. Thus when $\nu = (s^r)$,

$$f^{(s^r)}_{\lambda,\widehat{\lambda}} = \frac{\langle\, \lambda\,\rangle'_r}{\langle\, s^r\,\rangle'_r}. \tag{A.16}$$

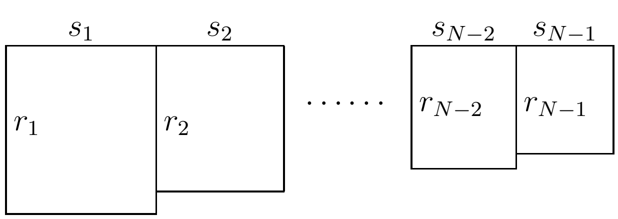
Note that $\langle\, \lambda\,\rangle'_r = \langle\, \widehat{\lambda}\,\rangle'_r$. For abbreviation, let $\langle\, s^r\,\rangle := \langle\, (s^r)\,\rangle$ and $\langle\, \widehat{n}\,\rangle := \langle\, \widehat{(n)}\,\rangle$, then we have

$$\frac{\langle\, s^r\,\rangle}{\langle\, n\,\rangle\langle\, \widehat{n}\,\rangle} f^{(s^r)}_{(n),\widehat{(n)}} = \prod_{\ell=0}^{n-1} \frac{(1-q^{s-\ell})(1-q^\ell t)}{(1-q^{\ell+1})(1-q^{s-\ell-1}t)}. \tag{A.17}$$

For the r variables $x := (x_1, \cdots, x_r)$, (eq. (VI.4.17) in [24]),

$$P_{\lambda+(s^r)}(x) = P_\lambda(x) P_{(s^r)}(x), \qquad P_{(s^r)}(x) = \prod_{j=1}^{r} x_i^s. \tag{A.18}$$

Let us denote the Young diagram decomposing into rectangles as $\lambda = \sum_{i=1}^{N-1}(s_i^{r_i})$, $r_i \ge r_{i+1}$, i.e., $\lambda' = (r_1^{s_1} r_2^{s_2} \cdots r_{N-1}^{s_{N-1}})$,

$\lambda =$ 

.

Then we have the following integral representation of the Macdonald polynomial [25]

$$P_\lambda(x) = \prod_{a=1}^{N-1} \frac{\langle\, \lambda^{(a)}\,\rangle}{r_a!\langle\, \lambda^{(a)}\,\rangle'_{r_a}} \cdot \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r_a} \frac{dz_j^a}{2\pi i z_j^a} (z_j^a)^{-s_a} \cdot \Pi(x, z^1) \prod_{a=1}^{N-1} \Pi(\overline{z^a}, z^{a+1}) \Delta^{Mac}(z^a) \tag{A.19}$$

with $z_i^N := 0$ and $\lambda^{(1)} := \lambda$, $\lambda^{(a)} := \sum_{i=a}^{N-1}(s_i^{r_i})$, i.e., $\lambda^{(a)\prime} = (r_a^{s_a} r_{a+1}^{s_{a+1}} \cdots r_{N-1}^{s_{N-1}})$. By replacing $\Delta^{Mac}(x)$ and z^a with $\Delta^{qW}(x)$ and $p^a z^a$, respectively, we also have [21]

$$\begin{aligned} P_\lambda(x) &= \widetilde{C}_\lambda^+ \oint \prod_{a=1}^{N-1} \prod_{j=1}^{r_a} \frac{dz_j^a}{2\pi i z_j^a} (z_j^a)^{-s_a} \cdot \Pi\left(x, pz^1\right) \prod_{a=1}^{N-1} \Pi\left(\overline{z^a}, pz^{a+1}\right) \Delta^{qW}(z^a) \\ &= \widetilde{C}_\lambda^+ \langle \alpha_{r,s}^+ | \exp\left\{ -\sum_{n>0} \frac{h_n^1}{1-q^n} \sum_{i=1}^M x_i^n \right\} |\chi_{r,s}^+\rangle, \quad \widetilde{C}_\lambda^+ := \prod_{a=1}^{N-1} \frac{p^{-ar_a s_a}\langle\, \lambda^{(a)}\,\rangle}{r_a!\langle\, \lambda^{(a)}\,\rangle''_{r_a}} \end{aligned} \tag{A.20}$$

with a singular vector $|\chi_{r,s}^+\rangle$ in (2.29).[5] Acting $\omega_-\omega_+\omega_{q,t}$ on (A.20) gives

$$P_{\lambda'}(x) = \widetilde{C}_\lambda^- \langle \alpha_{r,s}^- | \exp\left\{ -\sum_{n>0} \frac{h_n^1}{1-q^n} \sum_{i=1}^M (-q x_i)^n \right\} |\chi_{r,s}^-\rangle, \qquad \widetilde{C}_\lambda^- := \omega_-\omega_+ \frac{\widetilde{C}_\lambda^+}{\langle\, \lambda\,\rangle}. \tag{A.21}$$

Appendix B: Proof of (4.6)

Here we prove (4.6). We have the following formulas for the Young diagrams, which translate the summation in squares into that in lows [17]:

Lemma.

$$(1-q) \sum_{(i,j)\in\lambda} q^{j-1} t^{1-i} = \sum_{i=1}^{r} \left(1 - q^{\lambda_i}\right) t^{1-i}, \qquad r \geq \ell(\lambda), \tag{B.1}$$

$$(1-q) \sum_{(i,j)\in\mu} q^{\lambda_i - j} t^{\mu'_j - i} = \left[\sum_{i=1}^{r} \sum_{j=i}^{r} - t^{-1} \sum_{i=1}^{r} \sum_{j=i+1}^{r+1} \right] q^{\lambda_i - \mu_j} t^{j-i}, \qquad r \geq \ell(\mu). \tag{B.2}$$

In the following let us denote by (B.1)$(\lambda; q, t)$ and (B.2)(λ, μ) the equations (B.1) and (B.2), respectively. Using these we obtain

Lemma. *For any integer $r \geq \ell(\lambda), \ell(\mu)$,*

$$\begin{aligned} \sum_{i=1}^{r} \frac{q^{\mu_i} t^{r-i} - q^{s-\lambda_i} t^{i-1}}{1-q} &= \sum_{(i,j)\in(s^r)} q^{j-1} t^{i-1} - \sum_{(i,j)\in\lambda} q^{s-j} t^{i-1} - \sum_{(i,j)\in\mu} q^{j-1} t^{r-i} && \text{(B.3)} \\ &= \sum_{(i,j)\in\widehat{\mu}} q^{\widehat{\lambda}_i - j} t^{\widehat{\mu}'_j - i} - \sum_{(i,j)\in\lambda} q^{\mu_i - j} t^{\lambda'_j - i}. && \text{(B.4)} \end{aligned}$$

[5](A.20) can be written also by the Jackson integral [29].

Proof. First, $q^s\times$(B.1)$(\lambda;q^{-1},t^{-1})-t^{r-1}\times$(B.1)$(\mu;q,t)$ and $\sum_{i=1}^r (t^{r-1}-q^st^{i-1})/(1-q)=\sum_{i=1}^r\sum_{j=1}^s q^{j-1}t^{i-1}$ gives (B.3). Next, by (B.2)$(\widehat{\lambda},\widehat{\mu})$ and $\widehat{\lambda}_i:=s-\lambda_{r-i+1}$ with $\lambda_0:=0$,

$$
\begin{aligned}
(1-q)\sum_{(i,j)\in\widehat{\mu}} q^{\widehat{\lambda}_i-j}t^{\widehat{\mu}'_j-i} &= \left[\sum_{i=1}^{r}\sum_{j=i}^{r} - t^{-1}\sum_{i=1}^{r}\sum_{j=i+1}^{r+1}\right] q^{\widehat{\lambda}_i-\widehat{\mu}_j}t^{j-i}\\
&= \left[\sum_{i=1}^{r}\sum_{j=i}^{r} - t^{-1}\sum_{i=0}^{r-1}\sum_{j=i+1}^{r}\right] q^{\mu_i-\lambda_j}t^{j-i} \qquad\qquad \text{(B.5)}\\
&= \left[\sum_{i=1}^{r}\sum_{j=i}^{r} - t^{-1}\left(\sum_{i=1}^{r}\sum_{j=i+1}^{r+1} - \sum_{i=1}^{r}\sum_{j}\delta_{j,r+1} + \sum_{j=1}^{r}\sum_{i}\delta_{i,0}\right)\right] q^{\mu_i-\lambda_j}t^{j-i}.
\end{aligned}
$$

Thus (B.2)$(\widehat{\lambda},\widehat{\mu})-$(B.2)$(\mu,\lambda)$ gives (B.4). ◻

Note that $\widehat{\mu}'\neq\widehat{\mu'}$. From (B.3) and (B.4) and their $\mu=\widehat{\lambda}$ cases we have

$$\sum_{(i,j)\in(s^r)} q^{j-1}t^{1-i} = \sum_{(i,j)\in\lambda} q^{s-j}t^{i-r} + \sum_{(i,j)\in\widehat{\lambda}} q^{j-1}t^{1-i}, \qquad \text{(B.6)}$$

$$\sum_{(i,j)\in(s^r)} q^{j-1}t^{i-1} = \sum_{(i,j)\in\lambda} q^{s-j}t^{i-1} + \sum_{(i,j)\in\mu} q^{j-1}t^{r-i} + \sum_{(i,j)\in\widehat{\mu}} q^{\widehat{\lambda}_i-j}t^{\widehat{\mu}'_j-i} - \sum_{(i,j)\in\lambda} q^{\mu_i-j}t^{\lambda'_j-i}. \qquad \text{(B.7)}$$

For any equation $f(q,t)=0$ we define a mapping by $\exp\left\{-\sum_{n>0}\frac{1}{n}f(q^n,t^n)\right\}=0$. Acting this mapping on $u\times$(B.6) and (B.7) gives

$$\prod_{(i,j)\in(s^r)}(1-uq^{j-1}t^{1-i}) = \prod_{(i,j)\in\lambda}(1-uq^{s-j}t^{i-r})\prod_{(i,j)\in\widehat{\lambda}}(1-uq^{j-1}t^{1-i}), \qquad \text{(B.8)}$$

$$\prod_{(i,j)\in(s^r)}(1-q^jt^{i-1}) = \prod_{(i,j)\in\lambda}(1-q^{s-j+1}t^{i-1})\prod_{(i,j)\in\mu}(1-q^jt^{r-i})\frac{\prod_{(i,j)\in\widehat{\mu}}(1-q^{\widehat{\lambda}_i-j+1}t^{\widehat{\mu}'_j-i})}{\prod_{(i,j)\in\lambda}(1-q^{\mu_i-j+1}t^{\lambda'_j-i})}. \qquad \text{(B.9)}$$

Thus

$$\prod_{(i,j)\in(s^r)}\frac{(1-uq^{j-1}t^{1-i})}{(1-q^jt^{i-1})} = \prod_{(i,j)\in\lambda}\frac{(1-uq^{s-j}t^{i-r})}{(1-q^{s-j+1}t^{i-1})}\frac{\prod_{(i,j)\in\widehat{\lambda}}(1-uq^{j-1}t^{1-i})}{\prod_{(i,j)\in\mu}(1-q^jt^{r-i})}\frac{\prod_{(i,j)\in\lambda}(1-q^{\mu_i-j+1}t^{\lambda'_j-i})}{\prod_{(i,j)\in\widehat{\mu}}(1-q^{\widehat{\lambda}_i-j+1}t^{\widehat{\mu}'_j-i})}. \qquad \text{(B.10)}$$

When $\mu=\lambda$,

$$\prod_{(i,j)\in(s^r)}\frac{(1-uq^{j-1}t^{1-i})}{(1-q^jt^{i-1})} = \prod_{(i,j)\in\lambda}\frac{(1-uq^{s-j}t^{i-r})}{(1-q^{s-j+1}t^{i-1})}\frac{(1-q^{\mu_i-j+1}t^{\lambda'_j-i})}{(1-q^jt^{r-i})}\prod_{(i,j)\in\widehat{\lambda}}\frac{(1-uq^{j-1}t^{1-i})}{(1-q^{\widehat{\lambda}_i-j+1}t^{\widehat{\lambda}'_j-i})}. \qquad \text{(B.11)}$$

(A.8) and (A.5) completes the proof of (4.6).

Appendix C: Relation with Kaneko's integral formula

When $\beta \in \mathbb{N}$ we can use Kaneko's integral formula [30]. Let us define the following another kernel, which has the same $q=1$ limit with $\Delta^{qW}(z)$,

$$\begin{aligned}\Delta^{qH}(z) &:= \prod_{i<j} \exp\left\{-\sum_{n>0} \frac{1}{n} \frac{t^n - t^{-n}}{q^{\frac{n}{2}} - q^{-\frac{n}{2}}} q^{\frac{n}{2}} \frac{z_j^n}{z_i^n}\right\} \cdot \prod_{i=1}^{r} z_i^{(r+1-2i)\beta} \\ &= \prod_{i<j}\prod_{\ell\geq 1} \frac{1-q^\ell z_j/tz_i}{1-q^\ell t z_j/z_i} \cdot \prod_{i=1}^{r} z_i^{(r+1-2i)\beta}, \qquad |q|<1. \end{aligned} \tag{C.1}$$

In this section, we concentrate on the case of $\beta \in \mathbb{N}$ with $t = q^\beta$. Then

$$\begin{aligned}\Pi(z,w) &= \prod_{i,j}\prod_{\ell=0}^{\beta-1}(1-q^\ell z_i w_j)^{-1}, \\ \Delta^{qW}(z) &= \prod_{i<j}\prod_{\ell=0}^{\beta-1}(1-q^{-\ell}z_j/z_i)(1-q^\ell z_j/z_i)\cdot\prod_{i=1}^{r} z_i^{(r+1-2i)\beta} \\ &= \left(-q^{\frac{1-\beta}{2}}\right)^{\frac{\beta r(r-1)}{2}} \prod_{i<j}\prod_{\ell=0}^{\beta-1}(1-q^\ell z_i/z_j)(1-q^\ell z_j/z_i), \\ \Delta^{qH}(z) &= \prod_{i<j}\prod_{\ell=1-\beta}^{\beta}(1-q^\ell z_j/z_i)\cdot\prod_{i=1}^{r} z_i^{(r+1-2i)\beta} \\ &= \left(-q^{\frac{1-\beta}{2}}\right)^{\frac{\beta r(r-1)}{2}} \prod_{i<j}\prod_{\ell=0}^{\beta-1}(1-q^\ell z_i/z_j)(1-q^{\ell+1} z_j/z_i). \end{aligned} \tag{C.2}$$

Let $z := (z_1, \cdots, z_r)$ and $x := (x_1, \cdots, x_M)$. We have the following Kaneko's integral formula for the multivariate q-hypergeometric function in (4.13)

Lemma. [30][31]

$$\oint \prod_{j=1}^{r} \frac{dz_j}{2\pi i z_j} \prod_{\ell=0}^{a-1}(1-q^\ell z_j) \prod_{\ell=0}^{b-1}(1-q^{\ell+1}/z_j) \prod_{k=1}^{M}(1-z_j x_k)\cdot\Delta^{qH}(z) = {}_2\varphi_1^{(t,q)}\left[\begin{matrix} t^{-r}, q^b \\ q^{-a-1}t^{1-r}\end{matrix}; q^{-a}tx\right] c_{a,b,r}^{(q,t)} \tag{C.3}$$

with

$$c_{a,b,r}^{(q,t)} := \prod_{j=0}^{r-1} \frac{\prod_{\ell=0}^{a+b+j\beta}(1-q^{\ell+1})\prod_{\ell=0}^{(j+1)\beta}(1-q^{\ell+1})}{\prod_{\ell=0}^{a+j\beta}(1-q^{\ell+1})\prod_{\ell=0}^{b+j\beta}(1-q^{\ell+1})\prod_{\ell=0}^{\beta}(1-q^{\ell+1})}. \tag{C.4}$$

Note that the right hand side of (C.3) is summed over all Young diagrams λ with $\lambda_1 \leq r$ and $\ell(\lambda) \leq b$.

For any symmetric Laurent polynomial $f(z)$, we have

Lemma. [28]

$$\oint \prod_{j=1}^{r} \frac{dz_j}{2\pi i z_j} \cdot \Delta^{qW}(z) f(z) = r! \prod_{\ell=1}^{r} \frac{1-t}{1-t^{\ell}} \oint \prod_{j=1}^{r} \frac{dz_j}{2\pi i z_j} \cdot \Delta^{qH}(z) f(z). \tag{C.5}$$

This follows from

$$\Delta^{qH}(z) = \Delta^{qW}(z) \prod_{i<j} \frac{1 - t z_j/z_i}{1 - z_j/z_i} \tag{C.6}$$

and

$$\sum_{\omega \in \mathcal{S}_r} \omega \left(\prod_{i<j} \frac{1 - t z_j/z_i}{1 - z_j/z_i} \right) = \prod_{i=1}^{r} \frac{1-t^i}{1-t}. \tag{C.7}$$

Let $x := (x_1, \cdots, x_M)$. When

$$-\frac{t^{\frac{n}{2}} - t^{-\frac{n}{2}}}{q^{\frac{n}{2}} - q^{-\frac{n}{2}}} p_n = \sum_{k=1}^{M} x_k^n + \frac{q^{c\frac{n}{2}} - q^{-c\frac{n}{2}}}{q^{\frac{n}{2}} - q^{-\frac{n}{2}}} q^{-\frac{n}{2}(1+2s-c)}, \tag{C.8}$$

the partition function Z_2 in (4.4) is

$$Z_2 = \oint \prod_{j=1}^{r} \frac{dz_j}{2\pi i z_j} z_j^{-s} \prod_{\ell \geq 0} \frac{1 - q^{\ell - s} z_j}{1 - q^{\ell + c - s} z_j} \prod_{k=1}^{M} (1 - z_j x_k) \cdot \Delta^{qW}(z) =: Z_2^{(x)}. \tag{C.9}$$

When $c \in \mathbb{N}$, this reduces to

$$Z_2^{(x)} = (-1)^s q^{-\frac{s(s-1)}{2}} \oint \prod_{j=1}^{r} \frac{dz_j}{2\pi i z_j} \prod_{\ell=0}^{c-s-1} (1 - q^{\ell} z_j) \prod_{\ell=0}^{s-1} (1 - q^{\ell+1}/z_j) \prod_{k=1}^{M} (1 - z_j x_k) \cdot \Delta^{qW}(z). \tag{C.10}$$

Therefore, from (C.3) and (C.5) we obtain

Proposition.

$$\frac{Z_2^{(x)}}{Z_2^{(0)}} = {}_2\varphi_1^{(t,q)} \left[\begin{matrix} t^{-r}, q^s \\ q^{s-c-1} t^{1-r} \end{matrix} ; q^{s-c} t x \right]. \tag{C.11}$$

Note that the right hand side of (C.11) is nothing but that of (4.12) with $y = q^{-s}$ and $u = t^c$. Thus when $N = 2$ and $M = 1$, (C.11) coincides with the 5-dimensional $SU(2)$ Nekrasov partition function (5.16)

$$\frac{Z_2^{(x)}}{Z_2^{(0)}} = Z^{\text{inst}}(Q_1^{\pm}/v, Q_2^{\pm}/qv) \tag{C.12}$$

with (5.18).

Appendix D: Relation with Jackson integral

In this section we assume $0<q<1$. In this article $\oint \frac{dz}{2\pi i z} f(z)$ denotes the constant term in $f(z)$, i.e., $\oint \frac{dz}{2\pi i z}\sum_{n\in\mathbb{Z}} f_n z^n := \mathrm{CT}_{\{z\}}\sum_{n\in\mathbb{Z}} f_n z^n := f_0$. But to define the q-deformed β-ensemble (3.6), one can replace it by the Jackson integral which may have more natural $q\to 1$ limit. The Jackson integral is defined by

$$\int_0^1 \frac{d_q z}{z} f(z) := (1-q)\sum_{n\geq 0} f(q^n), \qquad \int_0^\infty \frac{d_q z}{z} f(z) := (1-q)\sum_{n\in\mathbb{Z}} f(q^n). \tag{D.1}$$

For a Laurent polynomial f, the relation between the Jackson integral and the constant term map is [32]

$$\mathrm{CT}_{\{z\}} f(z) = \lim_{\epsilon\to 0}\frac{1-q^\epsilon}{1-q}\int_0^1 \frac{d_q z}{z} z^\epsilon f(z). \tag{D.2}$$

For some special cases, one can calculate the partition function of the q-deformed β-ensemble (3.6) by using following formulas: For $\beta\in\mathbb{Z}_{\geq 0}$, $\Re(x)>0$ and $y\neq 0,-1,-2,\cdots$ [33],

$$\begin{aligned}&\int_0^1 \prod_{j=1}^r \frac{d_q z_j}{z_j} z_j^x \prod_{\ell\geq 0}\frac{1-q^{\ell+1}z_j}{1-q^{\ell+y}z_j}\cdot\prod_{i<j}\prod_{\ell=1}^{2\beta}(z_i-q^\ell z_j/t)\cdot\prod_{i,j}(1-z_i x_j)\\ &= t^{A_r}\prod_i \frac{\Gamma_q(i\beta+1)}{\Gamma_q(\beta+1)}\frac{\Gamma_q(x+(r-i)\beta)\Gamma_q(y+(r-i)\beta)}{\Gamma_q(x+y+(2r-i-1)\beta)}\,{}_2\varphi_1^{(t,q)}\left[\begin{matrix} t^{-r}, q^{-x}t^{1-r}\\ q^{-x-y}t^{2-2r}\end{matrix}; q^{-y}tx\right]\end{aligned} \tag{D.3}$$

with

$$A_r := \frac{r(r-1)}{2}x + \frac{r(r-1)(r-2)}{3}\beta. \tag{D.4}$$

And also for $\beta\in\mathbb{N}$ and $\Re(x)>-\lambda_r$ (Cor. 1.6 in [28]),[6]

$$\begin{aligned}&\int_0^1 \prod_{j=1}^r \frac{d_q z_j}{z_j} z_j^x \prod_{\ell=0}^{\beta}(1-q^{\ell+1}z_j)\cdot\prod_{i<j}\prod_{\ell=1}^{2\beta}(z_i-q^\ell z_j/t)\cdot\prod_{i,j}(1-z_i x_j)\cdot P_\lambda(z;q,t)\\ &= t^{A_r} P_\lambda(1,t,\cdots,t^{r-1};q,t)\prod_i \frac{\Gamma_q(i\beta+1)\Gamma_q(i\beta)}{\Gamma_q(\beta+1)}\frac{\Gamma_q(x+\lambda_i+(r-i)\beta)}{\Gamma_q(x+\lambda_i+(2r-i)\beta)}\\ &\times {}_{r+1}\varphi_r^{(t,q)}\left[\begin{matrix} t^{-r}, q^{-x-\lambda_1}t^{2-2r}, q^{-x-\lambda_2}t^{3-2r},\cdots, q^{-x-\lambda_r}t^{1-r}\\ q^{-x-\lambda_1}t^{1-2r}, q^{-x-\lambda_2}t^{2-2r},\cdots, q^{-x-\lambda_r}t^{-r}\end{matrix}; x\right].\end{aligned} \tag{D.5}$$

Note that If $\beta\notin\mathbb{Z}$ then for any regular function $f(z)$ in $|z|\leq 1$, using the pseudo constant $F(z)$ in (2.32), we can rewrite the contour integral to the Jackson integral

$$\oint_{|z|=1}\frac{dz}{2\pi i}F(z)z^{-\beta}f(z) = \sum_{n\geq 0} q^{n(1-\beta)} f(q^n)\mathrm{res}_{z=1}F(z) = \frac{1}{\Gamma_q(\beta)\Gamma_q(1-\beta)}\int_0^1 d_q z z^{-\beta} f(z). \tag{D.6}$$

[6] The first factor $\prod_{\ell=0}^{\beta}(1-q^{\ell+1}z_j)$ can be generalized to $\prod_{\ell\geq 0}(1-q^{\ell+1}z_j)/(1-q^{\ell+y}z_j)$ [34].

Appendix E: Four-dimensional case

E.1 $q \to 1$ limit

Here we give an example when $q = 1$, i.e., four-dimensional case [3]. Let us change the normalization of bosons by[7]

$$h_n^{i\ \mathrm{old}} = \sqrt{\frac{(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(t^{\frac{n}{2}} - t^{-\frac{n}{2}})}{n^2}}\, h_n^{i\ \mathrm{new}} = \frac{q^{\frac{n}{2}} - q^{-\frac{n}{2}}}{n}\sqrt{[\,\beta\,]_{q^n}}\, h_n^{i\ \mathrm{new}}, \qquad n \neq 0 \tag{E.1}$$

with $h_0^{i\ \mathrm{old}} = h_0^{i\ \mathrm{new}}$ and $Q_h^{i\ \mathrm{old}} = Q_h^{i\ \mathrm{new}}$ unchanged. Using the notation defined in the next subsection, let $q = e^{\hbar/\sqrt{\beta}}$, $h_n^{i\ \mathrm{new}} = (\vec{h}^i \cdot \vec{a}_n) + \mathcal{O}(\hbar)$ and $Q_h^{i\ \mathrm{new}} = (\vec{h}^i \cdot \vec{Q}) + \mathcal{O}(\hbar)$.[8] Then

$$p^{D_z} - \Lambda_i(zp^{1-i}) = -\hbar z^{\frac{N+1}{2}-i+1}(\alpha_0\partial_z + (\vec{h}^i \cdot \partial_z\vec{\phi}(z)))z^{-\frac{N+1}{2}+i} + \mathcal{O}(\hbar^2) \tag{E.2}$$

with $\alpha_0 := \sqrt{\beta} - 1/\sqrt{\beta}$ and $\partial_z := \frac{\partial}{\partial z}$. By letting $\hbar \to 0$ we obtain the four-dimensional case. Note that $\lim_{q\to 1}[\,n\,]_q = n$.

E.2 $\mathcal{W}_N$ algebra

Let $\{\vec{e}_i\}_{i=1}^N$ to be an orthonormal basis, i.e., $\vec{e}_i \cdot \vec{e}_j = \delta_{ij}$. The weight space of A_{N-1} is the hyper-surface perpendicular to $\sum_{i=1}^N \vec{e}_i$. The weights of the vector representation $\vec{h}^i$, the simple roots $\vec{\alpha}^a$ and the fundamental weights $\vec{\Lambda}^a$ for $i = 1, \cdots, N$ and $a = 1, \cdots, N-1$ are given by $\vec{h}^i := \vec{e}_i - \frac{1}{N}\sum_{j=1}^N \vec{e}_j$, $\vec{\alpha}^a := \vec{h}^a - \vec{h}^{a+1}$ and $\vec{\Lambda}^a := \sum_{i=1}^a \vec{h}^i$. Their inner-products are

$$\begin{aligned}
&(\vec{h}^i \cdot \vec{h}^j) = \delta^{i,j} - \frac{1}{N}, && (\vec{\alpha}^a \cdot \vec{\alpha}^b) = C^{ab} := 2\delta^{a,b} - \delta^{a-1,b} - \delta^{a+1,b}, \\
&(\vec{h}^i \cdot \vec{\alpha}^b) = B^{i,b} := \delta_{i,b} - \delta_{i-1,b}, && (\vec{\alpha}^a \cdot \vec{\Lambda}^b) = \delta^{a,b}, \\
&(\vec{h}^i \cdot \vec{\Lambda}^b) = A^{i,b} := \theta(i \le b) - \frac{b}{N}, && (\vec{\Lambda}^a \cdot \vec{\Lambda}^b) = (C^{-1})^{ab} = \min(a,b)\Big(1 - \frac{\max(a,b)}{N}\Big).
\end{aligned} \tag{E.3}$$

We define the boson field in the weight space by

$$\begin{aligned}
&\vec{\phi}(z) := \vec{Q} + \vec{a}_0 \log z - \sum_{n\neq 0} \tfrac{1}{n}\vec{a}_n z^{-n}, && [(\vec{\alpha}^a \cdot \vec{Q}), (\vec{\Lambda}^b \cdot \vec{Q})] = 0, \\
&[(\vec{\alpha}^a \cdot \vec{a}_n), (\vec{\Lambda}^b \cdot \vec{a}_m)] = n\delta^{a,b}\delta_{n+m,0}, && [(\vec{\alpha}^a \cdot \vec{a}_0), (\vec{\Lambda}^b \cdot \vec{Q})] = \delta^{a,b}.
\end{aligned} \tag{E.4}$$

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[7] Then $\frac{h_n^{i\ \mathrm{old}}}{q^{\frac{n}{2\ell}} - q^{-\frac{n}{2\ell}}} = \frac{1}{n}\frac{\sqrt{[\,\beta\,]_{q^n}}}{[\,1/\ell\,]_{q^n}} h_n^{i\ \mathrm{new}}$, $\frac{h_n^{i\ \mathrm{old}}}{t^{\frac{n}{2k}} - t^{-\frac{n}{2k}}} = \frac{1}{n}\frac{h_n^{i\ \mathrm{new}}}{[\,1/k\,]_{t^n}\sqrt{[\,\beta\,]_{q^n}}}$ and $\frac{(u^{\frac{n}{2}} - u^{-\frac{n}{2}})h_n^{i\ \mathrm{old}}}{(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(t^{\frac{n}{2}} - t^{-\frac{n}{2}})} = \frac{[\,\gamma\,]_{t^n}}{n}\sqrt{[\,\beta\,]_{q^n}}\, h_n^{i\ \mathrm{new}}$ for $u = t^\gamma$.

[8] $\mathcal{O}(\hbar)$ is a linear combination of $(\vec{h}^j \cdot \vec{a}_n)$'s and $(\vec{h}^j \cdot \vec{Q})$'s with $j = 1, \cdots, N$ [35].

Then for any vectors $\vec{u}$ and $\vec{v}$ in the weight space,

$$(\vec{u}\cdot\vec{\phi}(z))(\vec{v}\cdot\vec{\phi}(w)) = (\vec{u}\cdot\vec{v})\log(z-w) + :(\vec{u}\cdot\vec{\phi}(z))(\vec{v}\cdot\vec{\phi}(w)):. \tag{E.5}$$

From (E.2) the $\mathcal{W}_N$ generators $W^i_{c\ell}(z)$ can be defined by

$$\sum_{i=0}^{N} W^i_{c\ell}(z)(\alpha_0\partial_z)^{N-i} := :(\alpha_0\partial_z+(\vec{h}^1\cdot\partial_z\vec{\phi}(z)))(\alpha_0\partial_z+(\vec{h}^2\cdot\partial_z\vec{\phi}(z)))\cdots(\alpha_0\partial_z+(\vec{h}^N\cdot\partial_z\vec{\phi}(z))):. \tag{E.6}$$

Note that $W^i_{c\ell}(z) \neq \lim_{\hbar\to 0} W^i(z)$. $W^2_{c\ell}(z)$ is the Virasoro generator with the central charge $c = N-1-12\alpha_0^2\vec{\rho}^2$,

$$-W^2_{c\ell}(z) = \tfrac{1}{2}:(\partial\vec{\phi}(z)\cdot\partial\vec{\phi}(z)): + \alpha_0(\vec{\rho}\cdot\partial^2\vec{\phi}(z)) \tag{E.7}$$

where $\vec{\rho}$ is the half-sum of positive roots, $\vec{\rho} := \sum_{a=1}^{N-1}\vec{\Lambda}_a$, and $\vec{\rho}^2 = \frac{1}{12}N(N^2-1)$. Screening currents (2.15) and primary fields (2.21), (2.26) and (2.27) are now

$$S^a_\pm(z) := :e^{\pm\sqrt{\beta}^{\pm 1}(\vec{\alpha}^a\cdot\vec{\phi}(z))}:, \qquad V^a_\pm(z) := :e^{\mp\sqrt{\beta}^{\pm 1}(\vec{\Lambda}^a\cdot\vec{\phi}(z))}:, \tag{E.8}$$

$$V^a_{\ell+1,k+1}(z) := :e^{(-\ell\sqrt{\beta}+k/\sqrt{\beta})(\vec{\Lambda}^a\cdot\vec{\phi}(z))}:, \qquad V^a_\gamma(z) := :e^{-\gamma\sqrt{\beta}(\vec{\Lambda}^a\cdot\vec{\phi}(z))}:. \tag{E.9}$$

E.3 β-ensemble

The vertex operator in (3.1) is now

$$V_N := \prod_{a=1}^{N-1}\exp\left\{\sqrt{\beta}\sum_{n>0}\frac{1}{n}(\vec{\Lambda}^a\cdot\vec{a}_n)p_n^{(a)}\right\}. \tag{E.10}$$

Then, as (3.2), $\langle\alpha|V_N$ defines the isomorphism by

$$\sqrt{\beta}\; p_n^{(a)}\langle\alpha|V_N = \langle\alpha|V_N(\vec{\alpha}^a\cdot\vec{a}_{-n}), \qquad \frac{n}{\sqrt{\beta}}\frac{\partial}{\partial p_n^{(a)}}\langle\alpha|V_N = \langle\alpha|V_N(\vec{\Lambda}^a\cdot\vec{a}_n) \tag{E.11}$$

for $n>0$ and $\vec{\alpha}\langle\alpha|V_N = \langle\alpha|V_N\vec{a}_0$. Here $\langle\alpha|$ is an abbreviation of $\langle\vec{\alpha}|$. As (3.4) the vector $|S^+_{r,s}\rangle$ also defines another linear map by

$$(\vec{\Lambda}^a\cdot\vec{a}_n)|S^+_{r,s}\rangle = |S^+_{r,s}\rangle\sqrt{\beta}\sum_{k=1}^{r_a}(z^a_k)^n, \qquad n>0. \tag{E.12}$$

The partition function Z_N, the potential $W(z^a, z^{a+1})$ and the effective action S_{eff} in (3.6), (3.7) and (3.11), respectively, are now

$$Z_N := \oint\prod_{a=1}^{N-1}\prod_{j=1}^{r_a}\frac{dz^a_j}{2\pi i z^a_j}(z^a_j)^{-s_a}\exp\left\{\beta\sum_{n>0}\frac{1}{n}(z^a_j)^n p_n^{(a)}\right\}\cdot\Delta^{qW}(z^a)\Pi\left(\overline{z^a}, z^{a+1}\right),$$

$$W(z^a,z^{a+1}) := \sum_{i=1}^{r_a}\left(\beta\sum_{n>0}\frac{1}{n}(z_i^a)^n p_n^{(a)} - \beta\sum_{j=1}^{r_{a+1}}\log\left(1-\frac{z_j^{a+1}}{z_i^a}\right) - (s_a+1)\log z_i^a\right), \tag{E.13}$$

$$S_{\mathrm{eff}} := \sum_{a=1}^{N-1}\left(W(z^a,z^{a+1}) + 2\beta\sum_{i<j}\log\left(1-\frac{z_j^a}{z_i^a}\right) + \beta\sum_{i=1}^{r_a}(r_a+1-2i)\log z_i^a\right)$$

with $z^N := 0$ and

$$\Pi(z,w) := \prod_{i,j}\exp\left\{\beta\sum_{n>0}\frac{1}{n}z_i^n w_j^n\right\} = \prod_{i,j}(1-z_iw_j)^{-\beta}, \tag{E.14}$$

$$\Delta^{qW}(z) := \prod_{i<j}\exp\left\{-2\beta\sum_{n>0}\frac{1}{n}\frac{z_j^n}{z_i^n}\right\}\cdot\prod_{i=1}^{r} z_i^{(r+1-2i)\beta} = \prod_{i<j}(1-z_j/z_i)^\beta(z_i/z_j-1)^\beta \tag{E.15}$$

Z_N is written as (3.9) by the Jack polynomial $P_\lambda(x) := P_\lambda(x;\beta)$ defined in appendix E.6. The saddle point condition is $\frac{\partial S_{\mathrm{eff}}}{\partial z_k^a} = 0$ with

$$z_k^a\frac{\partial S_{\mathrm{eff}}}{\partial z_k^a} = \beta\sum_{n>0}(z_k^a)^n p_n^{(a)} + \beta\log\frac{\prod_{i<k}\left(1-z_k^a/z_i^a\right)^2\prod_{j=1}^{r_{a+1}}\left(1-z_j^{a+1}/z_k^a\right)}{\prod_{j>k}\left(1-z_j^a/z_k^a\right)^2\prod_{i=1}^{r_{a-1}}\left(1-z_k^a/z_i^{a-1}\right)} + ((r_a+1-2k)\beta - s_a - 1). \tag{E.16}$$

As (3.14) let us define $\partial_z\vec{\hat{\phi}}(z)$ and $\mathcal{W}^i_{c\ell}(z)$ by

$$\partial_z\vec{\hat{\phi}}(z) := z^{-1}\vec{\alpha} + \sqrt{\beta}\sum_{n>0}\sum_{b=1}^{N-1}\left(z^{n-1}\vec{\Lambda}^b p_n^{(b)} + z^{-n-1}\vec{\alpha}^b\frac{n}{\beta}\frac{\partial}{\partial p_n^{(b)}}\right), \tag{E.17}$$

$$\sum_{i=0}^{N}\mathcal{W}^i_{c\ell}(z)(\alpha_0\partial_z)^{N-i} := \boldsymbol{:}(\alpha_0\partial_z+(\vec{h}^1\cdot\partial_z\vec{\hat{\phi}}(z)))(\alpha_0\partial_z+(\vec{h}^2\cdot\partial_z\vec{\hat{\phi}}(z)))\cdots(\alpha_0\partial_z+(\vec{h}^N\cdot\partial_z\vec{\hat{\phi}}(z)))\boldsymbol{:} \tag{E.18}$$

and $\mathcal{W}^i_{c\ell}(z) =: \sum_{n\in\mathbb{Z}}\mathcal{W}^i_{c\ell n}z^{-n}$. Similarly, $\partial_z\vec{\tilde{\hat{\phi}}}(z)$ and $\widetilde{\mathcal{W}}^i_{c\ell}(z)$ are defined by replacing $\frac{n}{\beta}\frac{\partial}{\partial p_n^{(b)}}$ with $\sum_{k=1}^{r_j}(z_k^b)^n$. Then we have the $\mathcal{W}_N$ constraint $\mathcal{W}^a_{c\ell n}Z_N = 0$ and the loop equation $\left\langle\!\left\langle\, \widetilde{\mathcal{W}}^a_{c\ell n}\,\right\rangle\!\right\rangle = 0$ for $n>0$. The quantum spectral curve (3.23) is now

$$\left\langle\!\!\left\langle\,(\alpha_0\partial_z+(\vec{h}^1\cdot\partial_z\vec{\tilde{\hat{\phi}}}(z)))(\alpha_0\partial_z+(\vec{h}^2\cdot\partial_z\vec{\tilde{\hat{\phi}}}(z)))\cdots(\alpha_0\partial_z+(\vec{h}^N\cdot\partial_z\vec{\tilde{\hat{\phi}}}(z)))\,\right\rangle\!\!\right\rangle = 0. \tag{E.19}$$

For large r_a this reduces to (3.27) with $R=0$. When $p_n^{(a)} = \sum_{j=1}^{M_i}(x_j^{(a)})^n + (\gamma^{(a)}y^{(a)})^n$,

$$V_N = \prod_{a=1}^{N-1}\exp\left\{\sqrt{\beta}\sum_{n>0}\frac{1}{n}(\vec{\Lambda}^a\cdot\vec{a}_n)\left\{\sum_{j=1}^{M_i}(x_j^{(a)})^n + (\gamma^{(a)}y^{(a)})^n\right\}\right\} \tag{E.20}$$

is the positive mode part of $V^a_{\gamma^{(a)}}(1/y^{(a)})$ and $V^a_+(1/x_j^{(a)})$.[9]

[9]The Toda theory/W-gravity duality is discussed in [36].

E.4 $N = 2$ case

When $N = 2$, i.e., the Virasoro case, $\vec{\Lambda}^1 = \vec{h}^1 = -\vec{h}^2$, $\vec{\alpha}^1 = 2\vec{h}^1$, $(\vec{h}^1 \cdot \vec{h}^1) = A^{1,1} = 1/2$, $B^{1,1} = 1$ and $C^{1,1} = 2$. Let $p_n := p_n^{(1)}$ then the partition function Z_2 is now

$$Z_2(p) = \oint \prod_{j=1}^{r} \frac{dz_j}{2\pi i z_j} z_j^{-s} \exp\left\{\beta \sum_{n>0} \frac{1}{n} z_j^n p_n\right\} \cdot \Delta^{qW}(z) = \frac{r! \langle s^r \rangle_r''}{\langle s^r \rangle} P_{(s^r)}(x[p]). \tag{E.21}$$

As (4.11) the partition function $Z_2(p)$ substituting $p_n = \sum_i x_i^n + (\gamma y)^n$ is

$$\frac{Z_2\left(\sum_i x_i + \gamma y\right)}{Z_2(\gamma y)} = {}_2\varphi_1^{\beta}\left[\begin{matrix} -s, r\beta \\ 1 - s + (r-1-\gamma)\beta \end{matrix}; \frac{x}{y}\right]. \tag{E.22}$$

Here ${}_2\varphi_1^{\beta}\left[\begin{smallmatrix} a,b \\ c \end{smallmatrix}; x\right]$ is the multivariate hypergeometric function

$${}_2\varphi_1^{\beta}\left[\begin{matrix} a, b \\ c \end{matrix}; x\right] := \sum_{\substack{\lambda \\ \ell(\lambda) \le M}} P_\lambda(x;\beta) \prod_{(i,j)\in\lambda} \frac{(a+j-1+(1-i)\beta)(b+j-1+(1-i)\beta)}{(c+j-1+(1-i)\beta)(\lambda_i - j + 1 + (\lambda'_j - i)\beta)}. \tag{E.23}$$

If we substitute $\beta^n p_n = (-1)^{n-1}(\sum_i x_i^n + (\gamma y)^n)$ to $Z_2(p)$ then the right hand side ${}_2\varphi_1^{\beta}\left[\begin{smallmatrix} a,b \\ c \end{smallmatrix}; \frac{x}{y}\right]$ of (E.22) is replaced by ${}_2\varphi_1^{1/\beta}\left[\begin{smallmatrix} -a,-b \\ -c \end{smallmatrix}; \frac{x}{y}\right]$.

E.5 Four-dimensional Nekrasov partition function

Let $a = (a_1, \cdots, a_N)$ and $m = (m_1, \cdots, m_{2N})$ be sets of complex parameters. m_k corresponds to the mass of the fundamental matter. Then by (5.2) and (5.5),

$$\begin{aligned} Z^{\text{inst}}(a) &:= \sum_{\{\lambda_i\}} \prod_{i=1}^{N} (-\Lambda^2)^{N|\lambda_i|} \frac{\prod_{k=1}^{2N} N_{\lambda_i \bullet}\left(\frac{a_i + m_k}{\sqrt{\beta}} - \alpha_0/2\right)}{\prod_{j=1}^{N} N_{\lambda_i \lambda_j}\left(\frac{a_i - a_j}{\sqrt{\beta}}\right)}, \qquad \alpha_0 := \sqrt{\beta} - \frac{1}{\sqrt{\beta}}, \\ N_{\lambda\mu}(a) &:= N_{\lambda\mu}(a;\sqrt{\beta}) := (-1)^{|\lambda|+|\mu|} \\ &\times \prod_{(i,j)\in\lambda}\left(a + \frac{\lambda_i - j}{\sqrt{\beta}} + \sqrt{\beta}(\mu'_j - i + 1)\right) \prod_{(i,j)\in\mu}\left(a - \frac{\mu_i - j + 1}{\sqrt{\beta}} - \sqrt{\beta}(\lambda'_j - i)\right), \end{aligned} \tag{E.24}$$

which satisfies

$$\begin{aligned} N_{\lambda\mu}\left(a - \frac{\alpha_0}{2}; \sqrt{\beta}\right) &= N_{\mu\lambda}\left(a + \frac{\alpha_0}{2}; -\sqrt{\beta}\right) = N_{\mu'\lambda'}\left(a + \frac{\alpha_0}{2}; \frac{1}{\sqrt{\beta}}\right) \\ &= (-1)^{|\lambda|+|\mu|} N_{\mu\lambda}\left(-a - \frac{\alpha_0}{2}; \sqrt{\beta}\right). \end{aligned} \tag{E.25}$$

When $N = 2$, Z^{inst} coincides with the $M = 1$ case of the partition function Z_2

$$Z^{\text{inst}}\left(-m_1 + \tfrac{\beta-1}{2}, -m_2 + \tfrac{3\beta-1}{2}\right) = {}_2\varphi_1\left[\begin{matrix} a_2 + m_3 + \frac{1-\beta}{2}, a_2 + m_4 + \frac{1-\beta}{2} \\ a_2 - a_1 + 1 - \beta \end{matrix}; \Lambda^4\right] = \frac{Z_2(x + \gamma y)}{Z_2(\gamma y)} \tag{E.26}$$

with $s = -a_2 - m_3 - \frac{1-\beta}{2}$, $r\beta = a_2 + m_4 + \frac{1-\beta}{2}$, $\gamma\beta = a_1 + a_2 + m_3 + m_4 + 1 - \beta$ and $x/y = \Lambda^4$. Similarly, $Z^{\text{inst}}(-m_1 + \frac{\beta-1}{2}, -m_2 + \frac{\beta-3}{2})$ is given by changing the sign of the variables a, b and c of ${}_2\varphi_1\left[{a,b \atop c}; \Lambda^4\right]$ in (E.26).

E.6 Jack polynomial

Finally the Jack polynomials $P_\lambda(x) := P_\lambda(x;\beta)$ are defined by

$$\begin{aligned} HP_\lambda(x) &= \varepsilon_\lambda P_\lambda(x), \qquad \varepsilon_\lambda := \sum_{i=1}^{r} \lambda_i(\lambda_i + (r+1-2i)\beta), \\ H &:= \sum_{i=1}^{r} D_i^2 + \beta \sum_{i<j} \frac{x_i + x_j}{x_i - x_j}(D_i - D_j), \qquad D_x := x\frac{\partial}{\partial x} \end{aligned} \tag{E.27}$$

with a normalization condition $P_\lambda(x) = x_1^{\lambda_1} x_2^{\lambda_2} \cdots x_r^{\lambda_r} + \cdots$. Inner products $\langle f, g\rangle$ and $\langle f, g\rangle'_r$ are the same with (A.2) and (A.3), respectively, but $p_n^* := \frac{n}{\beta}\frac{\partial}{\partial p_n}$ and $\Delta^{Mac}(x) := \prod_{i\neq j}^{r} \exp\left\{-\beta \sum_{n>0} x_j^n / n x_i^n\right\} = \prod_{i\neq j}^{r}(1 - x_j/x_i)^\beta = (-1)^{\frac{r(r-1)}{2}\beta}\Delta^{qW}(x)$. The inner products of Jack polynomials are given by

$$\langle P_\lambda, P_\mu\rangle = \delta_{\lambda,\mu}\langle\, \lambda\,\rangle, \qquad \langle\,\lambda\,\rangle := \prod_{(i,j)\in\lambda} \frac{\lambda_i - j + 1 + (\lambda'_j - i)\beta}{\lambda_i - j + (\lambda'_j - i + 1)\beta}, \tag{E.28}$$

$$\langle P_\lambda, P_\mu\rangle'_r = \delta_{\lambda,\mu}\langle\,\lambda\,\rangle'_r, \qquad \frac{\langle\,\lambda\,\rangle'_r}{\langle\,\lambda\,\rangle} := \prod_{(i,j)\in\lambda} \frac{j - 1 + (r - i + 1)\beta}{j + (r-i)\beta} \prod_{k=1}^{r} \frac{\Gamma(k\beta)}{\Gamma(\beta)\Gamma((k-1)\beta+1)} \tag{E.29}$$

The specialization $p_n := \gamma$ with $\gamma \in \mathbb{C}$ is

$$P_\lambda\left(x[\,\gamma\,]\right) = \prod_{(i,j)\in\lambda} \frac{j - 1 + (\gamma + 1 - i)\beta}{\lambda_i - j + (\lambda'_j - i + 1)\beta}. \tag{E.30}$$

We have the following integral representation of the Jack polynomial [37]

$$\begin{aligned} P_\lambda(x;\beta) &= \widetilde{C}_\lambda^+ \langle \alpha_{r,s}^+| \exp\left\{-\sqrt{\beta}\sum_{n>0}\frac{h_n^1}{n}\sum_{i=1}^{M} x_i^n\right\}|\chi_{r,s}^+\rangle, \qquad \widetilde{C}_\lambda^+ := \prod_{a=1}^{N-1}\frac{(-1)^{\frac{r^{(a)}(r^{(a)}-1)}{2}\beta}\langle\,\lambda^{(a)}\,\rangle}{r_a!\langle\,\lambda^{(a)}\,\rangle'_{r_a}} \\ P_{\lambda'}\left(-x;\beta\right) &= \widetilde{C}_\lambda^- \langle \alpha_{r,s}^-| \exp\left\{-\sqrt{\beta}\sum_{n>0}\frac{h_n^1}{n}\sum_{i=1}^{M} x_i^n\right\}|\chi_{r,s}^-\rangle, \qquad \widetilde{C}_\lambda^- := \omega_-\omega_+\frac{\widetilde{C}_\lambda^+}{\langle\,\lambda\,\rangle} \end{aligned} \tag{E.31}$$

with $z_i^N := 0$.

Appendix F: Notation

Here we list up the notation of bosons.

h_n^i and Q_h^i $(i=1,2,\cdots,N)$: Fundamental (weight) bosons. $\sum_{i=1}^N p^{in} h_n^i = \sum_{i=1}^N Q_h^i = 0$.

Λ_n^a and Q_Λ^a $(a=1,2,\cdots,N-1)$: Weight bosons. $\Lambda_n^0 = \Lambda_n^N = Q_\Lambda^0 = Q_\Lambda^N = 0$.

α_n^a and Q_α^a $(a=1,2,\cdots,N-1)$: Root bosons.

F.1 Relations

	h_n^j	Λ_n^b	α_n^b
h_n^i		$= \displaystyle\sum_{b=1}^{N-1} B^{i,b}(p^n)\Lambda_n^b$	$= \displaystyle\sum_{b=1}^{N-1} A^{i,b}(p^n)\alpha_n^b,$
Q_h^i		$= Q_\Lambda^i - Q_\Lambda^{i-1}$	$= \displaystyle\left(\sum_{b=i}^{N-1} - \sum_{b=1}^{N-1}\frac{b}{N}\right) Q_\alpha^b,$
$\Lambda_n^a :=$	$\displaystyle\sum_{b=1}^{N-1}(B^{-1})^{a,b}(p^n) h_n^b$		$= \displaystyle\sum_{b=1}^{N-1}(C^{-1})^{a,b}(p^n)\alpha_n^b,$
$Q_\Lambda^a =$	$\displaystyle\sum_{b=1}^{a} Q_h^b$		$= \displaystyle\left(\sum_{b=1}^{a} b\frac{N-a}{N} + \sum_{b=a+1}^{N-1} a\frac{N-b}{N}\right) Q_\alpha^b,$
$\alpha_n^a :=$	$h_n^a - h_n^{a+1}$	$= \displaystyle\sum_{b=1}^{N-1} C^{a,b}(p^n)\Lambda_n^b,$	
$Q_\alpha^a =$	$Q_h^a - Q_h^{a+1}$	$= -Q_\Lambda^{a-1} + 2Q_\Lambda^a - Q_\Lambda^{a+1}.$	

Here

$$\begin{aligned}
A^{i,b}(p) &:= \frac{[\,N\theta(i\le b)-b\,]_p}{[\,N\,]_p} p^{\frac{b-N\theta(i>b)}{2}}, & (A^{-1})^{a,b}(p) &= \delta_{a,b} - \delta_{a+1,b} + p^{b-N}\delta_{a,N-1},\\
B^{i,b}(p) &:= p^{\frac{1}{2}}\delta_{i,b} - p^{-\frac{1}{2}}\delta_{i-1,b}, & (B^{-1})^{a,b}(p) &= p^{b-a-\frac{1}{2}}\theta(a\ge b),\\
C^{a,b}(p) &:= [\,2\,]_p\delta_{a,b} - p^{-\frac{1}{2}}\delta_{a-1,b} - p^{\frac{1}{2}}\delta_{a+1,b}, & (C^{-1})^{a,b}(p) &= \frac{[\,\min(a,b)\,]_p[\,N-\max(a,b)\,]_p}{[\,N\,]_p} p^{\frac{b-a}{2}}.
\end{aligned} \tag{F.1}$$

F.2 Commutation relations between bosons

$$[X_n^i, Y_m^j] = \frac{1}{n}(q^{\frac{n}{2}} - q^{-\frac{n}{2}})(t^{\frac{n}{2}} - t^{-\frac{n}{2}})\delta_{n+m,0}Z_n^{i,j}.$$

$X_n^i \backslash Y_{-n}^j$	h_{-n}^j	Λ_{-n}^b	α_{-n}^b
h_n^i	$\frac{[\,\delta_{ij}N-1\,]_{p^n}}{[\,N\,]_{p^n}} p^{\frac{n}{2}N\mathrm{sgn}(j-i)}$	$A^{i,b}(p^n)$	$B^{i,b}(p^n)$
Λ_n^a	$A^{j,a}(p^{-n})$	$(C^{-1})^{a,b}(p^n)$	$\delta_{a,b}$
α_n^a	$B^{j,a}(p^{-n})$	$\delta_{a,b}$	$C^{a,b}(p^n)$

F.3 Commutation relations between boson zero modes

$$[X_n^i, Q^j] = \delta_{n,0}Z^{i,j}.$$

$X_0^i \backslash Q^j$	Q_h^j	Q_Λ^b	Q_α^b
h_0^i	$\delta_{i,j} - \frac{1}{N}$	$\theta(i \leq b) - \frac{b}{N}$	$\delta_{i,b} - \delta_{i-1,b}$
Λ_0^a	$\theta(j \leq a) - \frac{a}{N}$	$\min(a,b)\left(1 - \frac{\max(a,b)}{N}\right)$	$\delta_{a,b}$
α_0^a	$\delta_{a,j} - \delta_{a+1,j}$	$\delta_{a,b}$	$2\delta_{a,b} - \delta_{a-1,b} - \delta_{a+1,b}$

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